Basic Gaussian Integrals

Theorem: For any complex α with positive real part and any complex β,

$$\int_{-\infty}^{+\infty} dx\, \exp\bigl(-\alpha(x+\beta)^2\bigr) \;=\; \sqrt{\frac{\pi}{\alpha}}\,. \tag{1}$$

Thanks to this theorem, the Fourier transform of a Gaussian wave packet is a Gaussian wave packet: For the x-apace wave packet

$$\Psi(x) \;=\; \Psi_0 \times e^{+ik_0x} \times \exp\left(-\frac{(x-x_0)^2}{2a^2}\right), \tag{2}$$

where a^2 can be complex as long as $\mathrm{Re}(a^2) > 0$, its k-space Fourier transform is

$$\widetilde{\Psi}(k) \;\stackrel{\text{def}}{=}\; \int dx\, e^{-ikx} \times \Psi(x) \;=\; \sqrt{2\pi}a\Psi_0 \times e^{-ix_0(k-k_0)} \times \exp\bigl(-\tfrac{1}{2}a^2(k-k_0)^2\bigr). \tag{3}$$

Indeed,

$$\begin{aligned}
&\int dx\, e^{-ikx} \times \Psi(x) \;= \\
&\quad= \int dx\, \Psi_0 \times \exp\left(-\frac{(x-x_0)^2}{2a^2} \;+\; ik_0x \;-\; ikx\right) \\
&\quad= \Psi_0 \exp(-ix_0(k-k_0)) \times \int dx\, \exp\left(-\frac{(x-x_0)^2}{2a^2} \;-\; i(k-k_0)(x-x_0)\right) \\
&\quad= \Psi_0 \exp(-ix_0(k-k_0)) \times \int dx\, \exp\left(-\frac{(x-x_0+ia^2(k-k_0))^2}{2a^2} \;-\; \frac{a^2(k-k_0)^2}{2}\right) \\
&\quad= \Psi_0 \exp(-ix_0(k-k_0)) \times \exp\bigl(-\tfrac{1}{2}a^2(k-k_0)^2\bigr) \times \\
&\qquad\qquad \times \int dx\, \exp\left(-\frac{(x-x_0+ia^2(k-k_0))^2}{2a^2}\right)
\end{aligned} \tag{4}$$

where the integral on the last line evaluates to $\sqrt{2\pi}a$ by the theorem (1).

Likewise, a k-space wave packet

$$\widetilde{\Psi}(k) \;=\; \widetilde{\Psi}_0 \times e^{-ix_0k} \times \exp\bigl(-\tfrac{1}{2}a^2(k-k_0)^2\bigr) \tag{5}$$

Fourier transforms to

$$\Psi(x) \;\stackrel{\rm def}{=}\; \int\frac{dk}{2\pi}\,e^{+ikx}\times\widetilde{\Psi}(x) \;=\; \frac{\widetilde{\Psi}_0}{\sqrt{2\pi}a}\times e^{+ik_0(x-x_0)}\times\exp\left(-\frac{(x-x_0)^2}{2a^2}\right). \tag{6}$$

Application to Dispersion

Let's apply the above math to the dispersion problem. Consider a 1-dimensional wave $\Psi(x,t)$ propagating through some dispersive media with a non-linear relation between wave number k and the frequency $\omega(k)$. Suppose at time $t_0=0$ we have a Gaussian wave packet

$$\Psi(x,0) \;=\; \Psi_0\times e^{+ik_0x}\times\exp\left(-\frac{(x-x_0)^2}{2a^2}\right) \tag{7}$$

whose width a is muck larger than the wavelength $2\pi/k_0$. Fourier transforming this wave packet to the k space, we get

$$\widetilde{\Psi}(k,0) \;=\; \sqrt{2\pi}a\Psi_0\times\exp\bigl(-\tfrac{1}{2}a^2(k-k_0)^2\bigr) \tag{8}$$

— another Gaussian packet of width $\Delta k\sim a^{-1}\ll k_0$. Consequently, when we evolve this packet to some future time $t>0$, we get

$$\widetilde{\Psi}(k,t) \;=\; e^{-i\omega(k)t}\times\widetilde{\Psi}(k,0) \;=\; \sqrt{2\pi}a\Psi_0\times e^{-i\omega(k)t}\times\exp\bigl(-\tfrac{1}{2}a^2(k-k_0)^2\bigr), \tag{9}$$

where for $|k-k_0|\ll k_0$ we may approximate

$$\omega(k) \;\approx\; \omega(k_0) \;+\; \frac{d\omega}{dk}\times(k-k_0) \;+\; \frac{1}{2}\frac{d^2\omega}{dk^2}\times(k-k_0)^2. \tag{10}$$

Consequently,

$$\widetilde{\Psi}(k,t) \;=\; \sqrt{2\pi}a\Psi_0 \times \exp\left(-\frac{a^2}{2}\,(k-k_0)^2 \;-\; \frac{it}{2}\frac{d^2\omega}{dk^2}\,(k-k_0)^2 \;-\; it\,\frac{d\omega}{dk}\,(k-k_0) \;-\; it\omega_0\right) \tag{11}$$

where the quadratic part of the exponent has a complex coefficient

$$C(t) \;=\; a^2 \;+\; it\,\frac{d^2\omega}{dk^2}\,. \tag{12}$$

In other words,

$$\widetilde{\Psi}(k,t) \;=\; \sqrt{2\pi}a\Psi_0 \times e^{-iv_gt(k-k_0)}e^{-i\omega_0 t} \times \exp\bigl(-\tfrac{1}{2}C(t)\times(k-k_0)^2\bigr) \tag{13}$$

where $v_g = (d\omega/dk)$ is the group velocity of the wave. The wave packet (13) has the form of eq. (8)for $x_0 = v_g t$ and $C(t)$ instead of a^2, so Fourier transforming it back to x space yields the packet of the form (2), namely

$$\Psi(x,t) \;=\; \frac{a\Psi_0}{\sqrt{C(t)}} \times e^{ik_0x-i\omega_0t} \times \exp\left(-\frac{(x-v_gt)^2}{2C(t)}\right). \tag{14}$$

We see that the wave pulse indeed moves to the right at the group velocity v_g, but at the same time its width increases with $C(t)$. Specifically, taking the magnitude^2 of the wave, we have

$$|\Psi(x,t)|^2 \;=\; \frac{a^2|\Psi_0|^2}{|C(t)|} \times \exp\left(-(x-v_gt)^2 \times \operatorname{Re}\left(\frac{1}{C(t)}\right)\right), \tag{15}$$

where

$$\operatorname{Re}\left(\frac{1}{C(t)}\right) \;=\; \frac{a^2}{a^4 \;+\; (\omega'')^2t^2}\,, \quad \omega'' \text{ being } \frac{d^2\omega}{dk^2}\,. \tag{16}$$

Thus, the time-dependent width of the wave pulse is

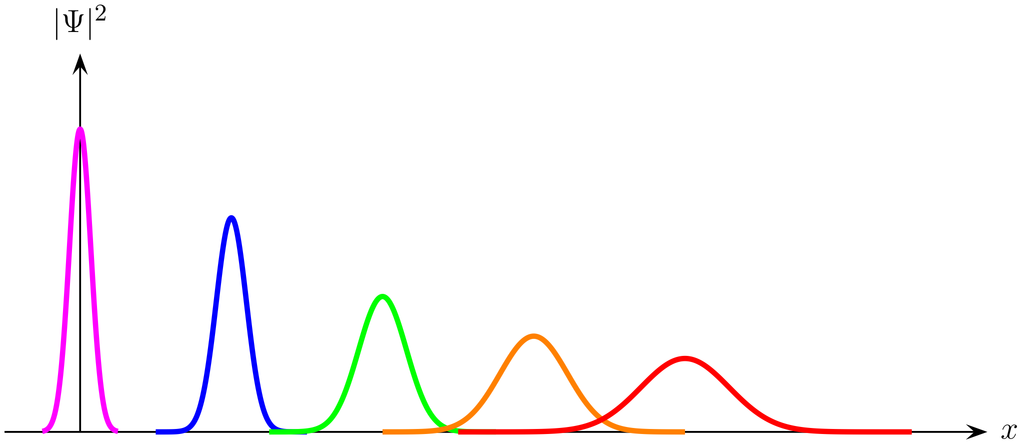
$$b(t) \;=\; \frac{1}{\sqrt{\operatorname{Re}(C^{-1})}} \;=\; \sqrt{a^2 \;+\; \frac{(\omega'')^2}{a^2}\,t^2} \tag{17}$$

— as promised, it increases with time for $\omega'' \neq 0$. Thus we see that $\omega'' \neq 0$ causes the wave pulse to disperse in space, and that's why the non-linear relation between k and ω is called the *dispersion*.

Besides the increasing pulse width, the peak amplitude of the wave pulse decreases with time as

$$\frac{a}{\sqrt{|C(t)|}} \;=\; \frac{a}{b(t)}\,. \tag{18}$$

To illustrate this effect, let me plot the magnitude of the pulse as a function of x at several times $t = 0, 1, 2, 3, 4$ (in units of $a^2/|\omega''|$):



The dispersion limits the pulse rate — and hence the information transfer rate — in long transmission lines, from $19^{\rm th}$ century telegraph cables to modern fiber optic cables. Indeed, whatever the initial pulse width a, by the time the pulse reaches the end of the line at time $T = L/v_g$, its width $b(T)$ must be shorter than the space interval between the pulses, or else we would not be able to resolve them from each other. In terms of the pulse rate

$$\nu \;=\; \frac{1}{\text{times between pulses}}\,, \tag{19}$$

we need

$$\frac{v_g}{\nu} \;>\; b(T) \tag{20}$$

and hence

$$\frac{v_g^2}{\nu^2} \;>\; b^2(T) \;=\; a^2 \;+\; \frac{(\omega'')^2T^2}{a^2}\,. \tag{21}$$

For a given travel time T, the RHS here is minimized for $a^2 = |\omega''|T$, thus even for this optimal width of the initial pulse, we need

$$\frac{v_g^2}{\nu^2} > 2|\omega''| \times T. \tag{22}$$

In other words, the pulse rate cannot be faster than

$$\nu_{\max} = \frac{v_g}{\sqrt{2|\omega''| \times T}}, \tag{23}$$

and that's why it's important to keep the dispersion ω'' in transmission lines as small as possible.

In terms of the refraction index $n(\omega)$, the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{c}{n + \omega(dn/d\omega)}, \tag{24}$$

hence

$$\begin{aligned} \omega'' \stackrel{\text{def}}{=} \frac{d^2\omega}{dk^2} &= \frac{d\omega}{dk} \times \frac{d}{d\omega}\left(\frac{d\omega}{dk}\right) = v_g \times \frac{dv_g}{d\omega} \\ &= \frac{v_g^3}{c} \times \frac{d}{d\omega}\left(-\frac{c}{v_g}\right) = -\frac{v_g^3}{c} \times \left(2\frac{dn}{d\omega} + \omega\,\frac{d^2n}{d\omega^2}\right), \end{aligned} \tag{25}$$

and therefore

$$\nu_{\max}^2 = \frac{v_g^2}{2|\omega''|} \times \left(\frac{1}{T} = \frac{v_g}{L}\right) = \frac{c}{2L} \Bigg/ \left|2\frac{dn}{d\omega} + \omega\,\frac{d^2n}{d\omega^2}\right|. \tag{26}$$

Thus, to maximize the pulse rate ν, we should endeavor to keep the refraction index $n(\omega)$ as constant as possible.