1. Let's start with an electrostatic problem. Suppose you are given the potential $\Phi(\mathbf{x})$ along a complete spherical surface of radius R relative to $\Phi(\infty) = 0$, and you know there are no electric charged anywhere outside that sphere. Then the potential at any point $\mathbf{y}$ outside the sphere can be found as

$$\Phi(\mathbf{y}) \;=\; \frac{\mathbf{y}^2 - R^2}{4\pi R} \iint\limits_{\text{sphere}} d^2\text{Area}(\mathbf{x})\, \frac{\Phi(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^3}\,. \tag{1}$$

Your task is to derive this formula in two different ways.

Let's start by solving the Laplace equation $\nabla^2\Phi(\mathbf{x}) = 0$ by separating variables in spherical coordinates: A general solution for the outside of a sphere subject to the asymptotic condition $\Phi(\infty) = 0$ has form

$$\Phi(r,\theta,\phi) \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} C_{\ell,m} \times r^{-\ell-1} \times Y_{\ell,m}(\theta,\phi) \tag{2}$$

for some constants complex $C_{\ell,m}$. The $Y_{\ell,m}(\theta,\phi)$ in this formula are the spherical harmonics you should be familiar with from your undergraduate quantum mechanics classes.

(0) If you are not familiar with eq. (2) and/or with the spherical harmonics, please read §3.1–3 and §3.5–6 of the Jackson's textbook.

(a) Determine the coefficients $C_{\ell,m}$ in the series (2) from the known potential at $r = R$, then write the potential outside the sphere as

$$\Phi(\mathbf{y}) \;=\; \iint\limits_{\text{sphere}} d^2\text{Area}(\mathbf{x})\, \Phi(\mathbf{x}) \times F(\mathbf{x},\mathbf{y}) \tag{3}$$

$$\text{for } F(\mathbf{x},\mathbf{y}) \;=\; \sum_{\ell,m} (\text{terms you need to calculate}). \tag{4}$$

(b) Sum up the series (4) and show that the integral (3) amounts to eq. (1). Here are some useful formulae:

$$\sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\mathbf{n}_y) Y^*_{\ell,m}(\mathbf{n}_x) \;=\; \frac{2\ell+1}{4\pi} \times P_\ell(\mathbf{n}_x \cdot \mathbf{n}_y), \tag{5}$$

$$\sum_{\ell=0}^{\infty} t^\ell P_\ell(c) \;=\; \frac{1}{\sqrt{1-2tc+t^2}} \qquad \text{for } |t|<1 \text{ and } |c|\le 1, \tag{6}$$

$$\sum_{\ell=0}^{\infty} (2\ell+1) t^\ell P_\ell(c) \;=\; \text{(find out from the previous formula).} \tag{7}$$

The other method to derive eq. (1) is to use the Green's function $G(\mathbf{x},\mathbf{y})$ for the Laplace equation outside the sphere subject to the Dirichlet boundary condition on the sphere itself, $G(\mathbf{x},\mathbf{y}) \equiv 0$ for $|\mathbf{x}| = R$, as well as the asymptotic condition $G(\mathbf{x},\mathbf{y}) \to 0$ for $|\mathbf{x}| \to \infty$.

(c) Use the image charge method for the outside of a grounded conducting sphere to show that

$$\begin{aligned} G(\mathbf{x},\mathbf{y}) &= \frac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|} - \frac{R/|\mathbf{y}|}{4\pi\,|\mathbf{x}-\frac{R^2}{|\mathbf{y}|^2}\mathbf{y}|} \\ &= \frac{1}{4\pi\sqrt{x^2+y^2-2xyc}} - \frac{R}{4\pi\sqrt{x^2y^2+R^4-2R^2xyc}} \end{aligned} \tag{8}$$

where $x = |\mathbf{x}|$, $y = |\mathbf{y}|$, and $c = \mathbf{n}_x \cdot \mathbf{n}_y$.

(d) Evaluate the normal derivative of this Green's function at the boundary.

(e) Finally, use the Green's function method (*cf.* textbook §1.10) to derive eq. (1).

2. Next, consider potential energies of electric multipoles in a slowly varying external electrostatic field $\mathbf{E}(\mathbf{x})$, and also the net forces and net torques on such multipoles. Note: in this problem, $\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$ is the *external* electric field created by some outside charges; it has nothing to do with the electric field due to the multipole in question itself.

As a warm-up exercise, consider a pure dipole, or rather a small charged body which has zero net charge, finite dipole moment $\mathbf{p}$, and negligibly small higher multipole moments $\mathcal{Q}_{ij}$, $\mathcal{O}_{ijk}, \ldots$

(a) Show that the potential energy of such a dipole located at point $\mathbf{x}_0$ in an external electric field is

$$U(\mathbf{x}_0,\mathbf{p}) \;=\; -\mathbf{p}\cdot\mathbf{E}(\mathbf{x}_0) \;=\; +\mathbf{p}\cdot\nabla\Phi(\mathbf{x}_0). \tag{9}$$

In classical mechanics, the net force $\mathbf{F}$ and the net torque acting on a body can be obtained from the variation of its potential energy under displacements and rotations of the body: For an infinitesimal displacement $\delta\mathbf{x}_0$ of the whole body and a rotation through infinitesimal angle $\delta\vec{\alpha}$, we should get

$$\delta U \;=\; -\mathbf{F}\cdot\delta\mathbf{x}_0 \;-\; \vec{\tau}\cdot\delta\vec{\alpha}. \tag{10}$$

Note: if you rotate the body relative to its own center $\mathbf{x}_0$, you get the torque $\vec{\tau}$ WRT that center rather than WRT the coordinate origin.

(b) Apply this rule to the electric dipole with potential energy (9) and show that

$$\mathbf{F} \;=\; \nabla(\mathbf{p}\cdot\mathbf{E}) \;=\; (\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{x}_0) \quad\text{and}\quad \vec{\tau} \;=\; \mathbf{p}\times\mathbf{E}(\mathbf{x}_0). \tag{11}$$

Now consider a pure quadrupole, or rather a small charged body which has zero net charge, zero dipole moment, a finite quadrupole moment $\mathcal{Q}_{ij}$, and negligibly small higher multipole moments.

(c) Show that the potential energy of such a quadrupole in a slowly varying external potential $\Phi(\mathbf{x})$ is

$$U \;=\; \tfrac{1}{3}\mathcal{Q}_{ij}\,\nabla_i\nabla_j\Phi(\mathbf{x}_0). \tag{12}$$

Hint: the external potential $\Phi(\mathbf{x})$ obeys the Laplace equation $\nabla_i\nabla_i\Phi = 0$.

(d) Find the net force and the net torque on the quadrupole from eq. (12) for the potential energy.

After all these preliminaries, consider a general charged body — which may have a net charge, a net dipole moment, a quadrupole moment, an octupole moment, *etc.*, *etc.* — in some slowly varying external potential.

(e) Generalize the above formulae and show that the potential energy of the body may be expanded in a series in derivatives of the external potential as

$$\begin{aligned} U(\mathbf{x}_), \text{moments}) \;&=\; \sum_{\ell=0}^{\infty} K_\ell \mathcal{M}^{(\ell)}_{i_1\cdots i_\ell} \times \nabla_{i_1}\cdots\nabla_{i_\ell}\Phi(\mathbf{x}_0) \\ &=\; q_{\text{net}}\Phi(\mathbf{x}_0) \;+\; p_i\nabla_i\Phi(\mathbf{x}_0) \;+\; \tfrac{1}{3}Q_{ij}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \\ &\qquad +\; \tfrac{1}{15}\mathcal{O}_{ijk}\nabla_i\nabla_j\nabla_k\Phi(\mathbf{x}_0) \;+\; \cdots \end{aligned} \tag{13}$$

where

$$K_\ell = \frac{1}{(2\ell-1)!!} = \frac{1}{(2\ell-1)(2\ell-3)\cdots(3)(1)}\,. \tag{14}$$

(Deriving eq. (12) for the coefficients K_n is an optional exercise. If you are short on time, skip it.)

Note: In eq. (13), the multipole moments of the body are taken WRT its center $\mathbf{x}_0$ rather than WRT the coordinate origin, thus

$$p_i = \iiint d^3\mathbf{x}\,\rho(\mathbf{x})(x-x_0)_i\,, \quad \mathcal{Q}_{ij} = \iiint d^3\mathbf{x}\,\rho(\mathbf{x})\left[\tfrac{3}{2}(x-x_0)_i(x-x_0)_j - \tfrac{1}{2}\delta_{ij}(\mathbf{x}-\mathbf{x}_0)^2\right], \tag{15}$$

etc., etc.

(f) Derive series similar to (13) for the net force and the net torque on the charged body.

3. Finally, consider the electrostatic potentials due to the multipole moments themselves.

As a warm-up exercise, let's start with the quadrupole-like potential

$$\Phi(\mathbf{x}) = \frac{Q_{ij}n_i^x n_j^x}{4\pi\epsilon_0\,|\mathbf{x}|^3} = \frac{Q_{ij}x_i x_j}{4\pi\epsilon_0\,|\mathbf{x}|^5} \tag{16}$$

for some would-be quadrupole moment tensor Q_{ij}, which you may assume to be symmetric — $Q_{ij} = Q_{ji}$ — but not necessarily traceless.

(a) Show that the potential (16) obeys the Laplace equation $\nabla^2\Phi = 0$ at $\mathbf{x} \neq 0$ if and only if the Q_{ij} tensor happens to be traceless, $Q_{ii} = 0$.

The point of this exercise is to explain why the quadrupole moment tensor of any charge distribution must be traceless. And it's defined as

$$Q_{ij} = \iiint d^3\mathbf{y}\,\rho(\mathbf{y})\Big(\tfrac{3}{2}y_i y_j - \tfrac{1}{2}\delta_{ij}\mathbf{y}^2\Big) \tag{17}$$

— with an extra $-\frac{1}{2}\delta_{ij}\mathbf{y}^2$ term inside the $(\cdots)$ — precisely to make it traceless.

Now let's generalize this result to higher multipoles: Consider a would-be 2^ℓ–pole potential

$$\Phi(\mathbf{x}) \;=\; \frac{\mathcal{M}^{(\ell)}_{i_1,\ldots,i_\ell}\, n^x_{i_1}\cdots n^x_{i_\ell}}{4\pi\epsilon_0\,|\mathbf{x}|^{\ell+1}} \;=\; \frac{\mathcal{M}^{(\ell)}_{i_1,\ldots,i_\ell}\, x_{i_1}\cdots x_{i_\ell}}{4\pi\epsilon_0\,|\mathbf{x}|^{2\ell+1}} \tag{18}$$

for some would-be 2^ℓ–pole moment tensor $\mathcal{M}^{(\ell)}_{i_1,\ldots,i_\ell}$. Again, assume this ℓ–index tensor to be totally symmetric under all possible permutations of its indices, but do not assume its tracelessness.

(b) Show that the potential (18) obeys the Laplace equation if and only if the tensor $\mathcal{M}^{(\ell)}$ happens to be traceless, $\mathcal{M}^{(\ell)}_{i_1,\ldots,i_{\ell-2},j,j} = 0$ for all $i_1,\ldots,i_{\ell-2} = 1,2,3$.

PS: The problem — as in telling you what to do — stops here. But if you are not familiar with traces of multi-index tensors, please read on.

Consider a tensor with $\ell > 2$ indices. To take its trace, choose any two indices of the tensor, and fix all the remaining $\ell-2$ indices. Restrict the chosen indices to equal values, and sum over all allowed values of an index — just as you would do for a two-index tensor. Repeat this procedure for all other values of the $\ell-2$ indices you are not tracing over — and this makes the trace into another tensor, but with $\ell-2$ indices instead of ℓ. For example, take a 3-index tensor T_{ijk} and take the trace over the first and the third index, thus $t_j = T_{iji}$ (implicit sum over $i = 1,2,3$). This trace t_j itself is a one-index tensor (*i.e.*, a vector).

For a non-symmetric tensor, we should specify which two of its ℓ indices we are tracing over. For example, for a non-symmetric 3–index tensor T_{ijk} we can trace over the first two indices and get $t^{(12)}_k = T_{iik}$, or over the last two indices and get $t^{(23)}_i = T_{ijj}$, or over the first and the third index and get $t^{(13)}_j = T_{iji}$. And in general, $t^{(12)}$, $t^{(23)}$, and $t^{(13)}$ would be three different 1-index tensors. However, for a totally symmetric ℓ-index tensor — such as the would-be multipole moment in eq. (18) — it does not matter which two indices we are tracing over. We may pick any two indices we like, and we would get exactly the same trace. Moreover, the trace itself would be a totally-symmetric $(\ell-2)$–index tensor.