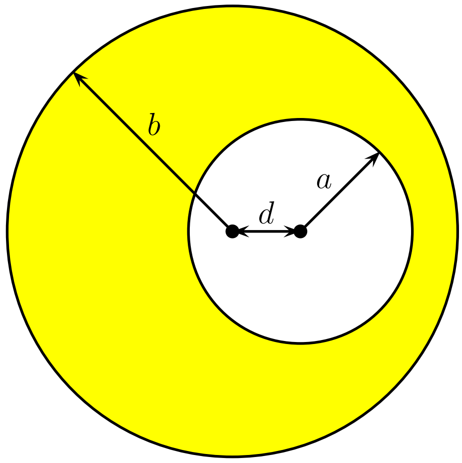
1. Consider a conductor in the shape of a long cylinder of radius b with a hole drilled in it. The hole is also cylindrical, or radius a, but it's not coaxial with the conductor itself. Instead, the hole's axis and the outer cylinder's axis are parallel but at non-zero distance d from each other. Take $a + d < b$ so that the hole does not reach the conductor's outer surface. Here is the cross-section:



   A current of uniform density $\mathbf{J}$ flows along this conductor in the direction $+z$ (out from the page into your face).

   Use the Ampere's Law and the linear superposition principle to find the magnetic field $\mathbf{B}$ — both the magnitude and the direction — inside the hole.

2. Consider a thin spherical shell of radius R with a uniform surface charge density σ. The sphere rotates about an axis through its center with angular velocity $\vec{\omega}$.

   (a) Write down the current density $\mathbf{j}(\mathbf{x})$ due to charges moving with the rotating sphere.

   (b) Calculate the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ inside and outside the rotating sphere.

   (c) Now consider two such rotating charged spheres, one inside the other. The two spheres are concentric, but they rotate around different, non-parallel axes, $\vec{\omega}_1 \nparallel \vec{\omega}_2$.

   Calculate the torque between the two spheres.

3. Consider the magnetic field inside a tightly wound solenoid of finite length L and finite radius R.

   (a) Using nothing but the rotational symmetry of the solenoid and the analyticity of the magnetic field as a function of the position $\mathbf{x}$, argue that in the cylindrical coordinates (z, s, ϕ),

$$B_z(z,s,\phi) \;=\; \sum_{n=0}^{\infty} \alpha_n(z) \times s^{2n} \;=\; \alpha_0(z) \;+\; \alpha_1(z)\,s^2 \;+\; \alpha_2(z)\,s^4 \;+\; \cdots, \tag{1}$$

$$B_s(z,s,\phi) \;=\; \sum_{n=0}^{\infty} \beta_n(z) \times s^{2n+1} \;=\; \beta_0(z)\,s \;+\; \beta_1(z)\,s^3 \;+\; \beta_2(z)\,s^5 \;+\; \cdots, \tag{2}$$

$$B_\phi(z,s,\phi) \;=\; 0, \tag{3}$$

   for some analytic functions $\alpha_n(z)$ and $\beta_n(z)$.

   (b) Next, use $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$ (inside the solenoid) to derive recursive relations between the functions $\alpha_n(z)$, $\beta_n(z)$ and their derivatives, and show that

$$\alpha_n(z) \;=\; \frac{(-1)^n}{2^{2n}\,(n!)^2}\left(\frac{d}{dz}\right)^{2n}\alpha_0(z), \qquad \beta_n(z) \;=\; \frac{(-1)^{n+1}}{2^{2n+1}\,(n+1)!\,n!}\left(\frac{d}{dz}\right)^{2n+1}\alpha_0(z). \tag{4}$$

In light of parts (a) and (b), given the magnetic field $B_z(z,0) = \alpha_0(z)$ on the axis of the solenoid as a function of z, the field off the axis obtains from it as a series

$$B_z(z,s) \;=\; \alpha_0(z) \;-\; \frac{s^2}{4}\,\alpha_0''(z) \;+\; \frac{s^4}{64}\,\alpha_0''''(z) \;+\; \cdots, \tag{5}$$

$$B_s(z,s) \;=\; -\frac{s}{2}\,\alpha_0'(z) \;+\; \frac{s^3}{16}\,\alpha_0'''(z) \;-\; \frac{s^5}{384}\,\alpha_0'''''(s) \;+\; \cdots. \tag{6}$$

   (c) Now approximate the winding of the solenoid as a cylindrical current sheet of density $K = IN/L$ and use the Biot–Savart–Laplace formula to show that the field on the cylinder's axis is

$$B_z(z,0) \;=\; \frac{\mu_0 IN}{L} \times \frac{1}{2}\left(\frac{(L/2)+z}{\sqrt{((L/2)+z)^2+R^2}} + \frac{(L/2)-z}{\sqrt{((L/2)-z)^2+R^2}}\right). \tag{7}$$

(d) Finally, consider a solenoid that's much longer than its radius and focus on the central region of $|z| = O(R) \ll L$. Estimate the derivatives of the on-axis field in this region and show that for $z = O(R)$ and any s between 0 and R,

$$B_z(z,s) \;=\; B_z(0,0) \times \left(1 + O\left(\frac{R^4}{L^4}\right)\right), \quad B_s(z,s) \;=\; B_z(0,0) \times O\left(\frac{R^4}{L^4}\right). \qquad (8)$$

Also, calculate the leading $O(R^4/L^4)$ terms in these formulae.