PHY–387 K. Problem set #3. Due February 14, 2019.

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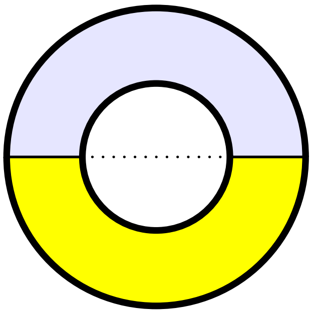
1. First, an easy electrostatic boundary problem. Two concentric metal spheres of respective radii a and b act as capacitor plates. Half of the space between the spheres (say, the lower hemisphere) is filled with a solid dielectric of dielectric constant ϵ while the other half is vacuum:



Note: the boundary between the dielectric and the vacuum lies in the same plane as the common center of the two spheres.

The voltage between the metal spheres is V.

   (a) Show that a radial **E** field between the spheres obeys the boundary conditions at the dielectric-vacuum interface.

   Then find the electric tension field **E** and the electric displacement field **D** everywhere between the spheres.

   (b) Find the surface charge densities on the metal spheres, and then the capacitance of this half-filled capacitor

2. Next, a reading assignment: §5.12 of Jackson's textbook about magnetic shielding by a spherical shell of high-permeability material. If you have trouble following the boundary condition for the magnetic scalar potential $\Psi(\mathbf{x})$ (which Jackson calls $\Phi_M(\mathbf{x})$, go back to the example of a dielectric sphere in external electric field in $4.4.

3. Now consider a wire loop $\mathcal{L}$ carrying steady current I. The loop $\mathcal{L}$ may have any size or shape, as long as it is closed. The magnetic field **H** generated by the current in this loop obtains from the scalar potential

$$\Psi(\mathbf{x}) \;=\; \frac{I}{4\pi}\,\Omega(\mathbf{x}) \tag{1}$$

where $\Omega(\mathbf{x})$ is the solid angle spanned by the loop $\mathcal{L}$ when viewed from the point $\mathbf{x}$.

By convention, $\Omega(\mathbf{x})$ is positive if the current in $\mathcal{L}$ viewed from point $\mathbf{x}$ appears to run clockwise, and negative if the current appears to run counterclockwise. To avoid a discontinuity when $\mathbf{x}$ is surrounded by the loop, $\Omega(\mathbf{x})$ should be analytically continued while $\mathbf{x}$ moves from one side of the loop to another. Such continuations makes Ω multivalued, with different values at the same point $\mathbf{x}$ differing by 4π, or more generally by $4\pi \times$ an integer.

(a) Show that

$$\Omega(\mathbf{x}) \;=\; \iint\limits_{\mathcal{S}} \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \cdot d^2\mathbf{area}(\mathbf{y}) \tag{2}$$

where $\mathcal{S}$ is a surface spanning the loop $\mathcal{L}$.

(b) Explain how eq. (2) leads to the sign convention for the $\Omega(\mathbf{x})$, and also how different surfaces $\mathcal{S}$ can yield values of $\Omega(\mathbf{x})$ which differ from each other by $4\pi \times$ an integer.

(c) Show that $\mathbf{H} = -\nabla\Psi$ for the scalar potential (1) agrees with the Biot–Savart–Laplace formula for the magnetic field of the current I in the loop $\mathcal{L}$.

Hint: prove and use

$$\nabla_y \times \left(\frac{(\mathbf{y}-\mathbf{x})\times\mathbf{c}}{|\mathbf{y}-\mathbf{x}|^3}\right) \;=\; (\mathbf{c}\cdot\nabla_y)\,\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \tag{3}$$

for $\mathbf{y} \neq \mathbf{x}$ and any constant vector $\mathbf{c}$.

4. Finally, an exercise on electrostatic energy. Take two large parallel vertical metal plates at small distance d between them, and immerse them part way into transformer oil with dielectric constant ϵ and *mass density* ρ. Connect the plates by wires to a battery or any other DC power supply of voltage V.

Show that this makes the oil between the plates rise to the height

$$h \;=\; \frac{(\epsilon-1)\epsilon_0 V^2}{2\rho g d^2} \tag{4}$$

relative to the oil outside the plates. ($g = 9.8$ N/kg is the gravitational field.)