1. In three space dimension, the retarded solution of the wave equation with an instant point source for example, a light flash at $\mathbf{x} = 0$ and $t = 0$ — is a spherical shell disturbance of radius $R = ct$ and zero thickness,

$$\Box\Psi(\mathbf{x}, t) \;=\; \delta(t)\delta^{(3)}(\mathbf{x}) \quad\Longrightarrow\quad \Psi(\mathbf{x}, t) \;=\; \frac{\delta(t - |\mathbf{x}|/c)}{4\pi|\mathbf{x}|}\,. \tag{1}$$

In spaces of other odd dimensions $d = 3, 5, 7, \ldots$ (but not $d = 1$), we have similar behavior, bun in even space dimensions $d = 2, 4, 6, \ldots$, the wave of an instant point source has a *wake* behind the light front. For example, in two space dimensions

$$\Box\Psi(\mathbf{x}, t) \;=\; \delta(t)\delta^{(2)}(\mathbf{x}) \quad\Longrightarrow\quad \Psi(\mathbf{x}, t) \;=\; \frac{2c\Theta(ct - |\mathbf{x}|)}{\sqrt{c^2t^2 - \mathbf{x}^2}} \tag{2}$$

where $\Theta$ is the step function.

(a) Derive the 2D wave (2) from the 3D wave generated by an instant *line* source.

In one space dimension, the disturbance spreads out at light speed, but then does not go away; instead,

$$\Box\Psi(x, t) \;=\; \delta(t)\delta(x) \quad\Longrightarrow\quad \Psi(x, t) \;=\; \frac{c}{2}\,\Theta(ct - |x|). \tag{3}$$

(b) Again, derive this 1D wave from the 3D wave generated by an instant source on an infinite plane.

2. In the Coulomb gauge $\nabla\cdot\mathbf{A} \equiv 0$, the potentials $\Phi$ and $\mathbf{A}$ obey

$$-\nabla^2\Phi(\mathbf{x}, t) \;=\; \frac{1}{\epsilon_0}\,\rho(\mathbf{x}, t), \quad \Box\mathbf{A}(\mathbf{x}, t) \;=\; \mu_0\mathbf{J}_T(\mathbf{x}, t), \tag{4}$$

where the transverse current $\mathbf{J}_T$ is

$$\mathbf{J}_T \;=\; \mathbf{J} \;+\; \nabla\left(\frac{-1}{\nabla^2}\,(\nabla\cdot\mathbf{J})\right), \quad i.e,, \quad \mathbf{J}_T(\mathbf{x}, t) \;=\; \mathbf{J}(\mathbf{x}, t) \;+\; \nabla_x \iiint d^3\mathbf{y}\,\frac{\nabla_y\cdot\mathbf{J}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\,. \tag{5}$$

As I've explained in class, the scalar potential $\Phi$ and part of the vector potential $\mathbf{A}$ respond instantaneously to the charges and currents, but these instantaneous terms cancel out from the electric and magnetic fields. In this exercise, we shall see how this works for a particularly simple source, namely an electric dipole $\mathbf{p}$ which turns up for just a moment and then turns back off,

$$\rho(\mathbf{x},t) \;=\; -\delta(t)\,(\mathbf{p}\cdot\nabla)\delta^{(3)}(\mathbf{x}), \quad \mathbf{J}(\mathbf{x},t) \;=\; \delta'(t)\,\mathbf{p}\,\delta^{(3)}(\mathbf{x}). \tag{6}$$

Note: derivatives of the delta functions are defined via integration by parts.

(a) As a warm-up trivial exercise, verify the continuity equation for the dipole flash (6) and calculate the scalar potential $\Phi(\mathbf{x},t)$ in the Coulomb gauge.

(b) Calculate the transverse current $\mathbf{J}_T(\mathbf{x},t)$ and show that

$$\begin{aligned}\mathbf{J}_T(\mathbf{x},t) \;&=\; \delta'(t)\left(\mathbf{p}\,\delta^{(3)}(\mathbf{x}) \;+\; \nabla(\mathbf{p}\cdot\nabla)\left(\frac{1}{4\pi r}\right)\right) &(7)\\ &=\; \delta'(t)\left(\frac{2}{3}\mathbf{p}\,\delta^{(3)}(\mathbf{x}) \;+\; \frac{3(\mathbf{n}\cdot\mathbf{p})\mathbf{n} \,-\, \mathbf{p}}{4\pi r^3}\right). &(8)\end{aligned}$$

(c) Next, prove a couple of lemmas you would need in the following parts:

$$\iiint\limits_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{z}\,\frac{\delta'(t-|\mathbf{z}|/c)}{|\mathbf{z}|}\,F(\mathbf{z}) \;=\; c^2\Theta(t)\left[\left(1 \,+\, r\frac{\partial}{\partial r}\right)\oint\!\!\!\oint d^2\Omega_n\,F(\mathbf{z}=r\mathbf{n})\right]_{@r=ct} \tag{9}$$

where $\Theta$ is the step-function, and

$$\frac{1}{4\pi}\oint\!\!\!\oint d^2\Omega_n\,\frac{1}{|\mathbf{x}+R\mathbf{n}|} \;=\; \frac{1}{\max(|\mathbf{x}|,R)}\,. \tag{10}$$

(d) Use the retarded Green's function to solve the wave equation $\Box\mathbf{A} \;=\; \mu_0\mathbf{J}_T$ for the vector potential under initial condition $\mathbf{A}(\mathbf{x},t<0)\equiv 0$. Show that

$$\begin{aligned}\text{for } t<|\mathbf{x}|/c, \quad \mathbf{A}(\mathbf{x},t) \;&=\; \mu_0 c^2\Theta(t)\,\nabla(\mathbf{p}\cdot\nabla)\left(\frac{1}{4\pi|\mathbf{x}|}\right)\\ &=\; \Theta(t)\,\frac{3(\mathbf{n}\cdot\mathbf{p})\mathbf{n} \,-\, \mathbf{p}}{4\pi\epsilon_0\,|\mathbf{x}|^3}\,.\end{aligned} \tag{11}$$

Hints: use eq. (7) rather that eq. (8) for the transverse current; after applying the retarded Green's function, change integration variable from $\mathbf{y}$ to $\mathbf{z}=\mathbf{y}-\mathbf{x}$, then turn $\partial/\partial y$ derivatives into $\partial/\partial x$ at fixed $\mathbf{z}$; use lemmas (9) and (10).

(e) Use vector potential (11) and the scalar potential you have computed in part (a) to verify that the electric and the magnetic fields do not propagate faster than light,

$$\text{for } t < c|\mathbf{x}|, \quad \mathbf{E}(\mathbf{x}, t) \;=\; \mathbf{B}(\mathbf{x}, t) \;=\; 0. \tag{12}$$

(f) Now calculate the vector potential $\mathbf{A}(\mathbf{x}, t)$ for $t \geq |\mathbf{x}|/c$, including the light front $t = |\mathbf{x}|/c$ itself.

(g) Finally, use the vector potential from part (f) to find the electric and magnetic fields.