

1. In class I have discussed electromagnetic energy, momentum, and related quantities for the vacuum. In a uniform linear medium we have similar formulae:

$$\text{power density:} \quad P = \mathbf{J} \cdot \mathbf{E}, \quad (1)$$

$$\text{force density:} \quad \mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}, \quad (2)$$

$$\text{energy density:} \quad u = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} + \frac{1}{2} \mathbf{H} \cdot \mathbf{D}, \quad (3)$$

$$\text{energy flux density:} \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}, \quad (4)$$

$$\text{momentum density:} \quad \mathbf{g} = \mathbf{D} \times \mathbf{B} = \frac{\epsilon \mu}{c^2} \mathbf{S}, \quad (5)$$

$$\text{stress tensor:} \quad T_{ij} = E_i D_j + H_i B_j - \frac{1}{2} \delta_{ij} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}). \quad (6)$$

- (a) Verify the local energy and momentum conservation laws

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} + P = 0, \quad (7)$$

$$\frac{\partial g_i}{\partial t} - \nabla_j T_{ij} + f_i = 0. \quad (8)$$

Hint: for a linear medium $E_i D_j = D_i E_j$ and $H_i B_j = B_i J_j$.

- (b) Which of the equations (1) through (6) — if any — should be modified in order to keep the conservation laws (7) and (8) in a linear but non-uniform media where ϵ and μ vary with \mathbf{x} (but not with time or frequency),

$$\mathbf{D}(\mathbf{x}, t) = \epsilon(\mathbf{x}) \epsilon_0 \mathbf{E}(\mathbf{x}, t), \quad \mathbf{B}(\mathbf{x}, t) = \mu(\mathbf{x}) \mu_0 \mathbf{H}(\mathbf{x}, t), \quad (9)$$

What is the physical meaning of this modification?

2. A thin toroidal coil of N turns has mean radius R , cross-section $a \ll R^2$, and carries steady current I . A point charge Q is placed at the center of the toroid. For simplicity, assume that electric field of this charge penetrates inside the coil without any distortion by the coil's wires.

Initially, the whole system — the charge, the coil, the battery and the wires providing the current — is at rest.

- Calculate the net momentum of the electromagnetic fields in the system.
 - Pick realistic values of the input parameters and calculate the resulting momentum. What kind of a mechanical system would have a similar momentum? A speeding bullet? A crawling ant? A proton in the LHC accelerator? Something else?
 - Let's turn off the current in the coil. The transient electric field induced by the dropping magnetic field imparts an impulse on the charge. Show that the net impulse given to the charge in this process is precisely the former momentum of the EM fields.
3. Consider the angular momentum of the electromagnetic field,

$$\mathbf{L}_{\text{EM}} = \iiint d^3\mathbf{x} \mathbf{x} \times \frac{1}{c^2} \mathbf{S}(\mathbf{x}) \quad (10)$$

where $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector.

- Show that for the static electric and magnetic fields, the angular momentum (10) can be written as

$$\mathbf{L}_{\text{EM}} = \frac{1}{c^2} \iiint d^3\mathbf{x} \Phi(\mathbf{x}) (\mathbf{x} \times \mathbf{J}(\mathbf{x}) - 2\mathbf{H}(\mathbf{x})). \quad (11)$$

- Now suppose magnetic monopoles exist. Put a static point electric charge Q at some distance a from a static monopole of magnetic charge M . Show that the angular momentum created by these electric and magnetic charges is

$$\mathbf{L}_{\text{EM}} = -\frac{\mu_0 M Q}{4\pi} \mathbf{n} \quad (12)$$

where \mathbf{n} is the unit vector pointing from the monopole towards the electric charge. Note: the magnitude of this angular momentum does not depend on the distance a between the electric charge and the monopole!

- (c) To verify eq. (12), let an electrically charge particle fly by a static monopole. For simplicity, assume that the only force acting on the particle is the Lorentz force due to the monopole's magnetic field, and that its motion is slow enough that we may use the quasistatic approximation to its electric field. Consequently, the net angular momentum of the system is the mechanical angular momentum of the particle plus the EM angular momentum (12),

$$\mathbf{L}_{\text{net}} = \mathbf{x} \times m\mathbf{v} - \frac{\mu_0 M Q}{4\pi} \mathbf{n}. \quad (13)$$

Verify the conservation of this angular momentum.

4. Finally, consider a 1D wave propagating through a linear and homogeneous but dispersive media with refraction index $n(\omega)$, *i.e.*, the phase velocity of a wave $v(\omega) = c/n(\omega)$. To allow for absorption, $n(\omega)$ may be complex rather than real.

- (a) Show that the most general solution of the dispersive wave equation is

$$\psi(x, t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left(A(\omega) \times \exp(+i\omega n(\omega)x/c) + B(\omega) \times \exp(-i\omega n(\omega)x/c) \right) \quad (14)$$

for some arbitrary complex functions $A(\omega)$ and $B(\omega)$.

- (b) Show that a real wave $\psi(x, t)$ requires $n(-\omega) = n^*(+\omega)$ as well as $A(-\omega) = A^*(+\omega)$ and $B(-\omega) = B^*(+\omega)$.
- (c) Suppose at $x = 0$ we observe ψ and its x derivative as functions of time. Show that in terms of these data

$$\begin{aligned} A(\omega) &= \int_{-\infty}^{+\infty} dt e^{+i\omega t} \left[\frac{1}{2} \psi(0, t) - \frac{ic}{2\omega n(\omega)} \frac{\partial \psi}{\partial x}(0, t) \right], \\ B(\omega) &= \int_{-\infty}^{+\infty} dt e^{+i\omega t} \left[\frac{1}{2} \psi(0, t) + \frac{ic}{2\omega n(\omega)} \frac{\partial \psi}{\partial x}(0, t) \right]. \end{aligned} \quad (15)$$