

1. Show that in the regime of normal dispersion — *i.e.*, at frequencies not too close to any of the resonances — the group velocity of the EM wave is always less than c . For simplicity, assume negligible magnetism $\mu \approx 1$ and use the low-density approximation to the dielectric constant,

$$\epsilon(\omega) \approx 1 + \frac{Ne^2}{\epsilon_0 m_e} \sum_i^{\text{resonances}} \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}. \quad (1)$$

Hint: show that $v_{\text{group}} < v_{\text{phase}}$ and $v_{\text{group}} \times v_{\text{phase}} < c^2$.

2. Consider charge density perturbations $\rho(\mathbf{x}, t)$ in a metal.
- (a) Fourier transform time-dependence of macroscopic EM fields and charge/current densities to frequency dependence. Use the transformed Maxwell equations — as well as

$$\mathbf{D}(\mathbf{x}, \omega) = \epsilon(\omega)\epsilon_0\mathbf{E}(\mathbf{x}, \omega) \quad \text{and} \quad \mathbf{J}_{\text{cond}}(\mathbf{x}, \omega) = \sigma(\omega)\mathbf{E}(\mathbf{x}, \omega) \quad (2)$$

where $\epsilon(\omega)$ and $\sigma(\omega)$ are the AC dielectric constant and conductivity — to show that $\rho(\mathbf{x}, \omega)$ obeys

$$\left(\sigma(\omega) - i\omega\epsilon(\omega)\epsilon_0\right)\rho(\omega, \mathbf{x}) = 0 \quad (3)$$

Hint: the net conduction + displacement current has zero divergence.

Drude–Lorentz formula tell us that in metals

$$\sigma(\omega) - i\omega\epsilon(\omega)\epsilon_0 = \frac{ne^2 f_0}{m_e^*} \frac{1}{\gamma_0 - i\omega} - i\omega\epsilon_b\epsilon_0 \approx \epsilon_b\epsilon_0 \left(\frac{\omega_p^2}{\gamma_0 - i\omega} - i\omega \right) \quad (4)$$

where $\gamma_0 = (1/\tau)$ is the rate at which the conduction electrons lose their average velocity vector to collisions with ions, and ω_p is the plasma frequency of the metal.

- (b) Solve eq. (3) for density perturbations in a metal with $\omega_p \gg \gamma_0$. Show that as a function of time rather than frequency, $\rho(t, \mathbf{x})$ oscillates in place with the plasma frequency ω_p while oscillation amplitude decays as $\exp(-\gamma_0 t)$.

3. In conducting materials, the EM waves attenuate with distance. For a specific example, consider a uniform material with dielectric constant ϵ and conductivity σ ; assume the frequency is low enough that ϵ and σ are real. Also assume negligible magnetism, $\mu = 1$.

The attenuating plane wave propagating in \mathbf{z} direction has general form

$$\mathbf{E}(x, y, z, t) = \vec{\mathcal{E}} \exp(ikz - \kappa z - i\omega t), \quad \mathbf{H}(x, y, z, t) = \vec{\mathcal{H}} \exp(ikz - \kappa z - i\omega t). \quad (5)$$

- (a) Write down formulae for k and κ as functions of ω . Also, relate the electric amplitude $\vec{\mathcal{E}}$ and the magnetic amplitude $\vec{\mathcal{H}}$ to each other.

Now consider a boundary between a conducting material and the vacuum. Suppose an EM wave comes from the vacuum side and hits the boundary head-on.

- (b) Calculate the reflectivity $R = |r|^2$ of the boundary.
(c) Show that for a good conductor

$$R \approx 1 - \frac{4\pi\delta}{\lambda_0} \quad (6)$$

where λ_0 is the wavelength of the EM wave in the vacuum and δ is the skin-depth of the current of the same frequency in the conductor.

- (d) As an example, find the reflectivity of sea water ($\sigma \approx 5\text{U/m}$) at an FM radio frequency $\omega = 2\pi \times 100\text{ MHz}$.