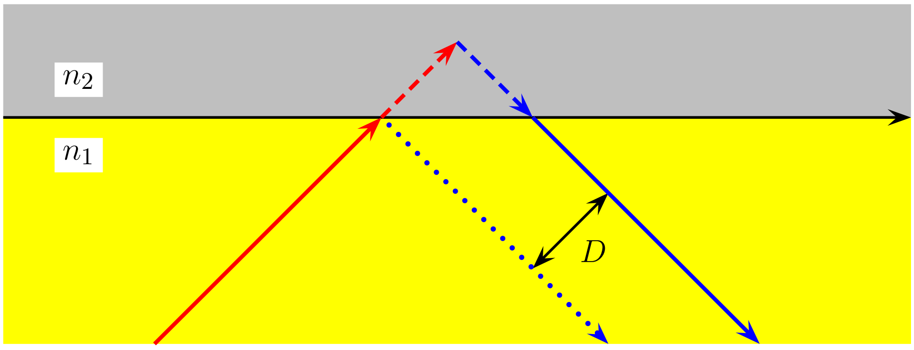
1. First, a reading assignment: §7.3 of the Jackson's textbook about reflection and refraction of the EM waves. Unlike my notes on the same subject, the texbook analysis allows for magnetic media on both sides of the boundary.

   ⋆ To check your understanding, rewrite the textbook formulae (7.39), (7.41), and (7.42) in terms of the wave impedance ratio Z'/Z of the two media.

2. Second, consider the Goos–Hänchen effect: In a total internal reflection, the reflected ray is displaced sideways relative to the incoming ray — as if it's reflected not from the boundary itself but from a small distance behind it.



(1)

The key to the Goos–Hänchen effect is the complex reflection coefficient

$$r(\alpha) \;=\; \exp(i\phi(\alpha)), \tag{2}$$

its magnitude in a total internal reflection is 1, but the phase depends on the incidence angle α.

(a) Suppose the incident wave has a finite but large width in the direction $\perp$ to the wave within the plane of incidence, for example

$$\begin{aligned}\mathbf{E}_i(x,y,z,t) \;&=\; \mathcal{E}_0\,\mathbf{e}_i\,\exp\bigl(ik_0(x\sin\alpha+z\cos\alpha)\,-\,i\omega t\bigr)\,\times\\ &\qquad\times\exp\left(-\frac{(x\cos\alpha-z\sin\alpha)^2}{2a^2}\right).\end{aligned} \tag{3}$$

for $a \gg (1/k_0)$. (In my notations, $\mathcal{E}_0$ is the overall amplitude of the wave and $\mathbf{e}$ its polarization vector.)

Fourier transform this wave to the **k** space, calculate the reflected wave (including its overall phase), then Fourier transform the reflected wave back to **x** space. Show that

$$\begin{aligned}\mathbf{E}_r(x,y,z,t) \;&=\; \mathcal{E}_0\,\mathbf{e}_r\,\exp\bigl(ik_0(x\sin\alpha - z\cos\alpha)\,-\,i\omega t\bigr)\times\\ &\qquad\times\int\frac{d\Delta k}{2\pi}\,A(\Delta k)\times\exp\bigl(i\Delta k((x\cos\alpha+z\sin\alpha)\,+\,i\phi(\mathbf{k}_0+\Delta\mathbf{k})\bigr)\end{aligned}\tag{4}$$

where $A(\Delta k) = \sqrt{2\pi}a\exp(-a^2\Delta k^2/2)$.

(b) Perform the Fourier integral in eq. (4) and show that

$$\begin{aligned}\mathbf{E}_r(x,y,z,t) \;&=\; \mathcal{E}_0\,\mathbf{e}_r\,\exp\bigl(ik_0(x\sin\alpha - z\cos\alpha)\,-\,i\omega t\bigr)\times\\ &\qquad\times e^{i\phi_0}\,\exp\left(-\frac{(x\cos\alpha+z\sin\alpha-D)^2}{2a^2}\right)\end{aligned}\tag{5}$$

for the displacement

$$D \;=\; -\frac{\partial\phi}{\partial\Delta\mathbf{k}_\perp} \;=\; -\frac{1}{k_0}\,\frac{\partial\phi}{\partial\alpha}\,.\tag{6}$$

(c) Analytically continue the Fresnel equations for the reflection coefficient r to the regime of total internal reflection and calculate its phase ϕ as a function of α. Note two different equations for the in-plane and normal-to-the-plane polarizations of the EM wave.

(d) Finally, put it all together and calculate the sideways displacement of the reflected wave. Show that

$$D_\perp \;=\; \frac{2}{k}\,\frac{\sin\alpha}{\sqrt{\sin^2\alpha-(n_2/n_1)^4}}\,,\tag{7}$$

$$D_\| \;=\; D_\perp\times\frac{1}{(1+(n_1/n_2)^2)\sin^2\alpha\,-\,1}\,.\tag{8}$$