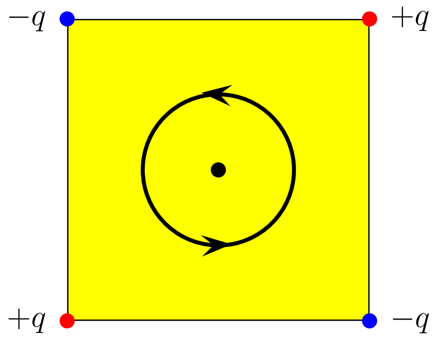
1. Four charges $\pm q$ sit at corners of a square of size $a \times a$, which rotates with frequency ω around the $\perp$ axis through the square's center.



   (a) Find the electric quadrupole moment tensor of this system. With what frequency does it rotate?

   (b) Find the angular distribution of the EM power radiated by the rotating quadrupole.

   (c) Find the net EM power radiated by the rotating quadrupole.

2. Consider an electric dipole with a general time-dependent dipole moment $\mathbf{d}(t)$. In a pure dipole approximation, the charge density and the current density of this system become

$$\rho(\mathbf{y},t) \;=\; -\big(\mathbf{d}(t)\cdot\nabla_y\big)\delta^{(3)}(\mathbf{y}), \quad \mathbf{J}(\mathbf{y},t) \;=\; \frac{d\,\mathbf{d}}{dt}\,\delta^{(3)}(\mathbf{y}) \tag{1}$$

   (a) Use the retarded Green's function to show that in the Landau gauge

$$\Phi(\mathbf{x},t) \;=\; \frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{n}}{r^2}\cdot\mathbf{d} + \frac{\mathbf{n}}{rc}\cdot\frac{d\,\mathbf{d}}{dt}\right]_{\text{ret}}, \qquad \mathbf{A}(\mathbf{x},t) \;=\; \frac{\mu_0}{4\pi r}\left[\frac{d\,\mathbf{d}}{dt}\right]_{\text{ret}}, \tag{2}$$

   where $\mathbf{d}$ and its time derivative should be evaluated at the retarded time $t_{\text{ret}} = t - r/c$.

   (b) Calculate the electric and the magnetic field for a general $\mathbf{d}(t)$.

(c) Now assume the harmonic time dependence $\mathbf{d}(t) = \mathbf{d}\exp(-i\omega t)$. Show that in this case

$$\begin{aligned} c\mathbf{B}(\mathbf{x},t) &= \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr-i\omega t}}{r}\left(1+\frac{i}{kr}\right)(\mathbf{n}\times\mathbf{d}), \\ \mathbf{E}(\mathbf{x},t) &= \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr-i\omega t}}{r}\left[\frac{i}{kr}\left(1+\frac{i}{kr}\right)\left(\mathbf{d}\,-\,3(\mathbf{n}\cdot\mathbf{d})\mathbf{n}\right)\,-\,\mathbf{n}\times(\mathbf{n}\times\mathbf{d})\right]. \end{aligned} \tag{3}$$

(d) Explain the long-distance and the short-distance limits of the EM fields (3).

3. Continuing the previous problem for a non-harmonic $\mathbf{d}(t)$.

(a) Calculate the long-distance limit of the Poynting vector and use it to show that the net power radiated by the time-dependent dipole moment is

$$P = \frac{Z_0}{6\pi c^2}\left[\frac{d^2\mathbf{d}}{dt^2}\right]^2_{\text{ret}}. \tag{4}$$

(b) As an example, consider a parallel-plate capacitor with plates of area A at distance b from each other. The capacitor is slowly charged to charge Q_0 and then is allowed to discharge through a resistor R, thus $Q(t) = Q_0\exp(-t/RC)$.

Find the net energy radiated by the capacitor while it discharges as a fraction of its initial energy $U_0 = Q_0^2/2C$.

(c) Calculate the actual numeric ratio U_{rad}/U_0 for $A = 10\text{ cm}\times 10\text{ cm}$, $b = 1\text{ mm}$, and $R = 10\ \Omega$.