1. Let's start with the relativistic velocity addition formula: Two velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ *in the same direction* add up according to

$$v_{1+2} = \frac{v_1 + v_2}{1 + (v_1 v_2 / c^2)}. \tag{1}$$

(a) Derive eq. (1) from two successive Lorentz transforms.

In 1851 Hyppolite Fizeau used interferometry to measure the speed of light in moving water or other liquids. He found that for light traveling in the same direction as the liquid or in the opposite direction, its speed is

$$u = \frac{c}{n} \pm \left(1 - \frac{1}{n^2}\right) \times v \tag{2}$$

where n is the refraction index of the liquid and v is its velocity.

(b) Derive eq. (2) from the relativistic velocity addition formula.

(c) Suppose the refraction index n of the liquid depends on the light frequency ω. Show that in this case, the phase velocity of light in the moving liquid becomes

$$u = \frac{c}{n} \pm \left(1 - \frac{1}{n^2} + \frac{\omega}{n}\frac{dn}{d\omega}\right) \times v. \tag{3}$$

2. The next problem is about the twin paradox. But first, consider a uniformly accelerating spaceship. That is, at any time $t > 0$ it has the same acceleration a relative to an inertial frame which at that moment has the same velocity as the ship.

(a) Show that the time τ aboard the ship, the time t on the planet where the ship has started from, and the velocity v of the ship relative to that planet are related to each other as

$$\frac{at}{c} = \sinh\left(\frac{a\tau}{c}\right), \quad \frac{v}{c} = \tanh\left(\frac{a\tau}{c}\right) = \frac{at}{\sqrt{c^2 + (at)^2}}. \tag{4}$$

Hint: use the relativistic velocity addition formula (1).

(b) Find the distance of the ship from its starting point as a function of t and as a function of τ. Also, show that a light signal sent from the starting point at any time $t > c/a$ will never catch up with the ship as long as it keeps accelerating.

Now consider a round trip from Earth to the (possibly habitable) planet Gliese 667 Cc, about 23.62 light years from Earth. For the astronaut's convenience, the ship accelerates at constant rate $a = g = 9.80\ \mathrm{m/s^2}$ from Earth to the mid-point, then decelerates at the same rate until it stops at the destination. It spends a year at the Gliese 667 Cc planet, then flies back in the same manner: accelerates at the constant rate $a = g$ to the midpoint, then decelerates to stop at the Earth.

(c) If a crew member has a twin who stayed on Earth and the trip started on their 21$^{\mathrm{st}}$ birthday, how old would be each twin by the time the ship comes back to Earth?

3. Finally, let's go back to relativistic velocity addition, but this time we allow for different directions of velocities.

(a) An inertial frame of reference K' moves at velocity $\mathbf{u}$ relative to another inertial frame K. A particle has velocity $\mathbf{v}'$ relative to the K' frame. Show that its velocity $\mathbf{v}$ relative to K frame is

$$\mathbf{v} = \left(1 + \frac{\mathbf{u}\cdot\mathbf{v}'}{c^2}\right)^{-1}\left(\mathbf{u} + \mathbf{v}_{\|} + \frac{1}{\gamma_u}\mathbf{v}'_{\perp}\right) \tag{5}$$

where $\mathbf{v}'_{\|}$ and $\mathbf{v}'_{\perp}$ are the components of the $\mathbf{v}'$ vector parallel and perpendicular to the relative velocity $\mathbf{u}$ of the two frames.
Note: eq. (5) is not symmetric WRT $\mathbf{v}' \leftrightarrow \mathbf{u}$, unless the $\mathbf{v}'$ and $\mathbf{u}$ vectors happen to be parallel to each other.

(b) Verify that eq. (5) is consistent with the speed of light universality. That is, show that if the particle in question is a photon and $|\mathbf{v}'| = c$, then also $|\mathbf{v}| = c$, although the directions of the $\mathbf{v}'$ and $\mathbf{v}$ vectors are generally different.

Now consider an accelerating particle and the relation between the acceleration vectors $\mathbf{a}'$ and $\mathbf{a}$ relative to the frames K' and K. Please allow for completely general directions of the vectors $\mathbf{a}'$, $\mathbf{v}'$, and $\mathbf{u}$.

(c) Show that

$$\mathbf{a}_{\parallel} \;=\; \frac{\left(1 \;-\; \dfrac{\mathbf{u}^2}{c^2}\right)^{3/2}}{\left(1 \;+\; \dfrac{\mathbf{u}\cdot\mathbf{v}'}{c^2}\right)^{3}}\;\mathbf{a}'_{\parallel}\,, \tag{6}$$

$$\mathbf{a}_{\perp} \;=\; \frac{\left(1 \;-\; \dfrac{\mathbf{u}^2}{c^2}\right)}{\left(1 \;+\; \dfrac{\mathbf{u}\cdot\mathbf{v}'}{c^2}\right)^{3}}\left[\mathbf{a}'_{\perp} \;+\; \frac{\mathbf{u}}{c^2}\times(\mathbf{a}'\times\mathbf{v}')\right], \tag{7}$$

where $\mathbf{a}_{\parallel}$ and $\mathbf{a}_{\perp}$ are the components of the acceleration vector $\mathbf{a}$ respectively parallel and perpendicular to the relative velocity $\mathbf{u}$ of the two frames, and ditto for the $\mathbf{a}'_{\parallel}$ and $\mathbf{a}'_{\perp}$ components of the $\mathbf{a}'$ acceleration vector.