1. Consider Ohm's Law in a moving conducting medium. The Lorentz covariant form of the Ohm's Law is

$$J^\mu \; - \; \frac{J^\nu u_\nu}{c^2}\, u^\mu \;\; = \;\; \frac{\sigma}{c}\, F^{\mu\nu} u_\nu \tag{1}$$

where u^μ is the 4–velocity of the medium,

$$U^\mu \;\; = \;\; \frac{dx^\mu}{d\tau}, \quad u^0 \;\; = \;\; \gamma c, \quad \mathbf{u} \;\; = \;\; \gamma \mathbf{v}. \tag{2}$$

(a) Verify that in the rest frame of the conducting medium eq. (1) becomes the usual Ohm's Law equation $\mathbf{J}' \;=\; \sigma \mathbf{E}'$ where σ is the conductivity and primes denote the rest-frame quantities.

(b) Now, for a general inertial frame, spell out eq. (1) in 3D vector notations. Note: you should get two separate equations, one for $\mu = 0$ and one for $\mu = i = 1, 2, 3$.

(c) Next, let's focus on the lab frame in which the medium moves at velocity $\mathbf{v} \neq 0$. Suppose you are given both the EM fields $\mathbf{E}$ and $\mathbf{B}$ and the electric charge density ρ, all in the lab frame. Show that the electric current density in that frame is (in Gauss units)

$$\mathbf{J} \;\; = \;\; \rho \mathbf{v} \; + \; \gamma \sigma \left[\mathbf{E} \; + \; \frac{\mathbf{v}}{c} \times \mathbf{B} \; - \; \frac{\mathbf{v}(\mathbf{v} \cdot \mathbf{E})}{c^2} \right]. \tag{3}$$

(d) Finally, suppose we know that the moving medium is electrically neutral in its rest frame, $\rho' = 0$. Show that in the lab frame

$$\mathbf{J} \;\; = \;\; \gamma \sigma \left(\mathbf{E} \; + \; \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad \rho \;\; = \;\; \frac{\gamma \sigma}{c^2} \left(\mathbf{v} \cdot \mathbf{E} \right). \tag{4}$$

2. Consider a point particle of charge q moving at constant velocity vector $\mathbf{v}$; in worldline terms

$$x^{\mu}_{\text{charge}}(\tau) \;=\; u^{\mu} \times \tau. \tag{5}$$

In covariant form, the EM fields created by this particle are

$$F^{\mu\nu}(x) \;=\; \frac{q}{c}\,\frac{\left(x^{\mu}u^{\nu} - x^{\nu}u^{\mu}\right)}{R'^{3}} \tag{6}$$

where

$$R'^{2} \;=\; \frac{(x \cdot u)^{2}}{c^{2}} \;-\; (x \cdot x). \tag{7}$$

(a) Verify that for a particle at rest, the electric components of the $F^{\mu\nu}$ tensor (6) comprize the usual Coulomb field of the point particle (in Gauss units) while the magnetic components are absent, $\mathbf{B} = 0$.

(b) Show that for a moving charge

$$R'^{2} \;=\; \gamma^{2}(x_{\parallel} - vt)^{2} \;+\; \mathbf{x}_{\perp}^{2}. \tag{8}$$

In other words, eq. (7) is the covariant way of writing the distance from the moving charge in that charge's rest frame, *cf.* eq. (54) of my notes.

(c) Finally, spell out the EM field tensor (6) for a moving charge in 3D terms and check that the $\mathbf{E}$ and $\mathbf{B}$ fields agree with eqs. (52) of my notes, specifically

$$\mathbf{E}(\mathbf{x}, t) \;=\; \frac{\gamma Q(\mathbf{x} - \mathbf{v}t)}{R'^{3}}, \quad \mathbf{B}(\mathbf{x}, t) \;=\; \vec{\beta} \times \mathbf{E}(\mathbf{x}, t). \tag{9}$$

3. Lorentz symmetries combine the energy density, the energy flux density, the momentum density, and the stress tensor into a symmetric Lorentz tensor $T^{\mu\nu} = +T^{\nu\mu}$ called the *stress-energy tensor*. For the electromagnetic fields in the vacuum, the stress-energy tensor can be written in covariant form as

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\mu\alpha} g_{\alpha\beta} F^{\beta\nu} + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{10}$$

(Gauss units).

(a) Spell out the components of the stress-energy tensor in 3D terms and explain the physical meaning of the components.

(b) Use Maxwell equations (in the covariant form) to show that

$$\partial_\mu T^{\mu\nu} = \frac{1}{c} J_\mu F^{\mu\nu}. \tag{11}$$

(c) Spell out eq. (11) in 3D terms — separately for $\nu = 0$ and for $\nu = j = 1, 2, 3$ — and explain the physical meaning of these equations.

4. Consider photo-production of pions via the following process:

$$p + \gamma \rightarrow \Delta^+ \rightarrow n + \pi^+ \tag{12}$$

— a proton absorbs a photon and becomes a Δ^+ resonance, which then decays into a neutron and a positive pion. At both stages of this process, the net energy-momentum is conserved,

$$P^\mu(p) + P^\mu(\gamma) = P^\mu(\Delta) = P^\mu(n) + P^\mu(\pi), \tag{13}$$

while for each of the particles here

$$P^\mu((i)P_\mu(i) = c^2 M^2(i). \tag{14}$$

Specifically, the rest masses of particles involved in this process are as follows: Proton, $M_p \approx 938$ MeV/c^2; pion, $M_{\pi^+} \approx 139$ MeV/c^2; Δ^+ resonance, $M_{\Delta^+} \approx 1226$ MeV/c^2; neutron, $M_n \approx 939$ MeV/c^2 (you may approximate $M_n \approx M_p$); and photon, exactly zero.

(a) In the center-of-mass frame — which is also the frame in which the Δ^+ resonance is at rest — what are the energies of the neutron and the pion?

(b) In the lab frame — the frame in which the initial proton is at rest — what should the photon's energy be in order to make the Δ^+ resonance?

(c) For the lab frame, derive the relation between the energy of the pion and the direction of its velocity. Give numerical values for the maximal and minimal values of pion's energy and find out whether the pion can move backwards (relative to the incident photon)?

(d) Now consider the frame where the target proton moves at a relativistic velocity. If a photon collides with this proton "head on", it takes a lower photon energy to make a Δ^+ resonance. For an extremely high energy cosmic ray proton, even a photon from the cosmic microwave background can make a Δ^+. This is known as the GZK (Greisen–Zatsepin–Kuzmin) effect which makes it difficult for the ultra high energy protons to fly more than a few megaparsecs through the intergalactic space.

While the energy spectrum of the microwave background peaks about $\frac{2}{3} \times 10^{-3}$ eV, the photons with 3 times higher energy $\hbar\omega = 2 \cdot 10^{-3}$ eV are numerous enough to act as GZK obstacles for the ultra high energy protons. How much energy does a proton needs to collide with such a photon and make a Δ^+ resonance?