1. Consider once again a point charge moving at constant velocity $\mathbf{v}$; in worldline terms, $x_c^\mu(\tau) = \tau \times u^\mu$. The EM fields produced by this charge were calculated in my notes on covariant electrodynamics by Lorentz-boosting the Coulomb field of a point charge at rest, and in your previous homework set#12 you should have verified the covariant form of the EM field tensor,

$$F^{\mu\nu}(x) \;=\; \frac{q}{c} \times \frac{x^\mu u^\nu \,-\, x^\nu u^\mu}{\left[c^{-2}(x\cdot u)^2 \,-\, (x\cdot x)\right]^{3/2}}\,. \tag{1}$$

On the other hand, in my notes on radiation by moving charges we used Liénard–Wiechert *retarded* potentials to show that for a non-accelerating point charge

$$F^{\mu\nu}(x) \;=\; \left[\frac{qc^2(n^\mu u^\nu \,-\, n^\nu u^\mu)}{R^2(n\cdot u)^3}\right]_{\rm ret} \tag{2}$$

where R and $n^\mu = (1,\mathbf{n})$ should be evaluated at the retarded time $(x^0 - R)/c$ rather than the observer time X^0/c.

Your task is to reconcile the two fomulae (1) and (2).

2. A particle of charge q, mass m, and a *non-relativistic* initial velocity $v_0 \ll c$ collides head-on with a repulsive central potential $V(r)$, and bounces back. While the particle is decelerated — and then accelerated on its way back — by the potential, it emits EM radiation.

   (a) Show that the total energy emitted by the particle is

$$\Delta W \;=\; \frac{4q^2}{3m^2c^3} \times \int\limits_{R_0}^{\infty} \frac{dr}{\sqrt{v_0^2 \,-\, (2/m)V(r)}} \times \left|\frac{dV}{dr}\right|^2 \tag{3}$$

   where R_0 is the closest approach of the particle to the potential center. (That is, the point where $V(r) = \frac{1}{2}mv_0^2$.)

(b) For the special case of the Coulomb potential $V(r) = Qq/r^2$, show that

$$\Delta W = \frac{8}{45}\frac{q}{Q}\frac{mv_0^5}{c^3}, \tag{4}$$

or in terms of the initial kinetic energy $E_0 = \frac{1}{2}mv_0^2$ of the particle,

$$\frac{\Delta W}{E_0} = \frac{16}{45}\frac{q}{Q}\left(\frac{v_0}{c}\right)^3. \tag{5}$$

3. Consider a relativistic particle of charge q decelerating to stop at uniform rate during a very short time interval Δt. That is, for $t < 0$ the particle moves at constant velocity $\vec{\beta}_0 c$, for $0 < t < \delta t$, the particle have constant acceleration $\mathbf{a} = -\vec{\beta}_0 c/\Delta t$ (in the lab frame) in the direction directly opposite the velocity, at $t = \Delta t$ the particle comes to stop, and for $t > \Delta t$ it remains at rest. While the particle is decelerating, it emits EM radiation.

(a) Show that the net radiant energy emitted per unit of solid angle is

$$\frac{dE}{d\Omega} = \frac{q^2\beta_0^2}{16\pi c\Delta t} \times \frac{\sin^2\theta(2-\beta_0\cos\theta)(2-2\beta_0\cos\theta+\beta_0^2\cos^2\theta)}{(1-\beta_0\cos\theta)^4}. \tag{6}$$

(b) Show that for ultra-relativistic particles with initial $\gamma_0 \gg 1$, this radiant energy distribution becomes strongly concentrated in the forward cone of opening angle $\sim 1/\gamma_0$. Also, show that within this cone

$$dE = \frac{q^2\gamma_0^4}{\pi c\Delta t} \times \frac{\xi^3\, d\xi\, d\phi}{(1+\xi^2)^4} \quad \text{for} \quad \xi = \gamma_0 \times \theta, \tag{7}$$

and the total radiation energy emitted is

$$E = \frac{q^2\gamma^4}{6c\Delta t}. \tag{8}$$