Beyond the Long Wavelength Limit

Thus far, we have studied EM radiation by oscillating charges and current confined to a volume of linear size much smaller than the wavelength $\lambda = 2\pi c/\omega$. In these notes, we shall work out EM radiation by an antenna of length comparable to λ or even longer than λ.

But let's start with the general theory. For any harmonically oscillating current $\mathbf{J}(\mathbf{x}, t) = \mathbf{J}(\mathbf{x}) \exp(-i\omega t)$, the vector potential of the radiation emitted by this current is (in the Landau gauge)

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \iiint d^3\mathbf{y} \, \frac{\exp(ik|\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} \, \mathbf{J}(\mathbf{y}). \tag{1}$$

This formula is valid for all $\mathbf{x}$, but let's focus on the far zone where

$$r = |\mathbf{x}| \gg \lambda \quad \textbf{and} \quad r \gg \text{antenna size.} \tag{2}$$

Note: for a small antenna it's the first condition here which defines the far zone, but for a large antenna the second condition takes over. And thanks to this second condition, we have $|\mathbf{x}| \gg |\mathbf{y}|$ for all $\mathbf{y}$ contributing to the integral (1), hence

$$|\mathbf{x} - \mathbf{y}| \approx r - \mathbf{n} \cdot \mathbf{y} \tag{3}$$

and therefore

$$\frac{\exp(ik|\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} \approx \frac{\exp(ikr)}{r} \times \exp(-ik\mathbf{n} \cdot \mathbf{y}). \tag{4}$$

Consequently, in the far zone we have a spherically divergent EM wave

$$\mathbf{A}(\mathbf{x}, t) \approx \mu_0 \, \frac{\exp(ikr - i\omega t)}{r} \, \mathbf{f}(\mathbf{n}), \tag{5}$$

where

$$\mathbf{f}(\mathbf{n}) = \frac{1}{4\pi} \iiint d^3\mathbf{y} \, \mathbf{J}(\mathbf{y}) \exp(-ik\mathbf{n} \cdot \mathbf{y}). \tag{6}$$

The electric and the magentic fields in the far zone — and hence the EM power radiated in the direction $\mathbf{n}$ — follow from eq. (5) in the usual way, just as we learned last week:

$$\mathbf{H}(\mathbf{x}, t) = ik \, \frac{\exp(ikr - i\omega t)}{r} \, \big(\mathbf{n} \times \mathbf{f}(\mathbf{n})\big), \tag{7}$$

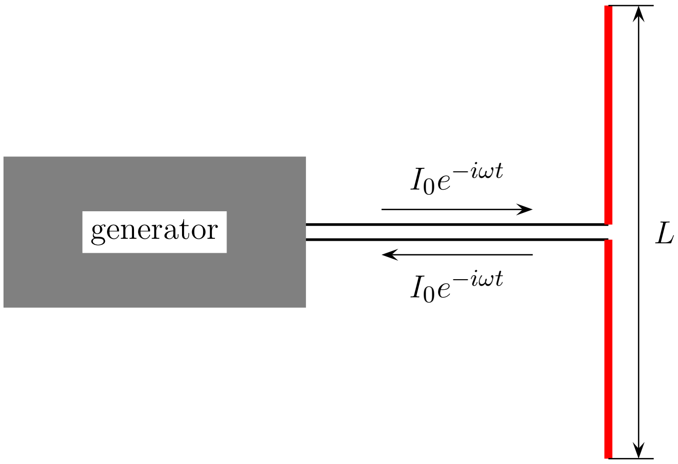
$$\mathbf{E}(\mathbf{x},t) \;=\; ikZ_0\,\frac{\exp(ikr-i\omega t)}{r}\,\bigl(-\mathbf{n}\times(\mathbf{n}\times\mathbf{f}(\mathbf{n}))\bigr), \tag{8}$$

$$\frac{dP}{d\Omega} \;=\; \frac{Z_0k^2}{2}\,\bigl\|\mathbf{n}\times\mathbf{f}(\mathbf{n})\bigr\|^2. \tag{9}$$

But the devil hides in details, or in our case in eq. (6) for the direction-dependent amplitude $\mathbf{f}(\mathbf{n})$: For a short antenna of size $\ll\lambda$, we may expand the amplitude $\mathbf{f}(\mathbf{n})$ into a series of electric and magnetic multipoles, but for longer antennas such expansion does not work. Instead, we need to evaluate the integral (6) in all its awful glory...

Example: Center-Fed Linear Antenna

As an example of a large antenna, consider a center-fed linear antenna of size $L\sim\lambda$:



For simplicity, let's assume the antenna's diameter is negligibly thin compared to the antenna's length, so we may approximate the current in the antenna as

$$\mathbf{J}(x,y,z,t) \;=\; I(z)\delta(x)\delta(y)\exp(-i\omega t)\mathbf{n}_3 \tag{10}$$

where $\mathbf{n}_3$ is a unit vector along the z axis direction. For such a current, eq. (6) becomes

$$\mathbf{f}(\mathbf{n}_x) \;=\; \frac{\mathbf{n}_3}{4\pi}\int\limits_{-L/2}^{+L/2} dz\,I(z)\,\exp\bigl(-ik\mathbf{n}_x\cdot\mathbf{z}\;=\;-ikz\cos\theta\bigr). \tag{11}$$

Next consider the z dependence of the current $I(z)$ in the antenna. To calculate this current from the first principles, we need equations and boundary conditions for the EM fields immediately outside the antenna, which we should convert into an equation of the $I(z)$, and then solve that equation. I shall address some of these issues at the end of these notes. But for the moment, let me simply *assume* that the current — or rather the disturbances of the current — propagates along the antenna at the speed of light. Consequently, the harmonically oscillating current forms a standing wave obeying

$$\left(\frac{d^2}{dz^2} + k^2\right) I(z) \;=\; 0, \tag{12}$$

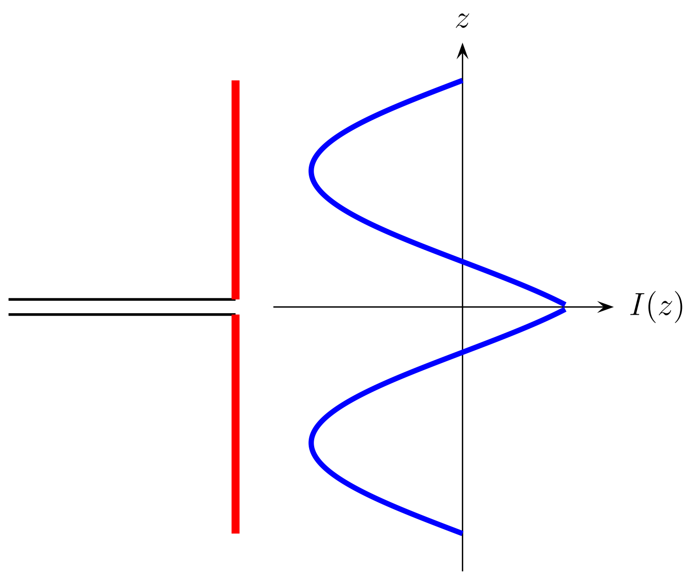
except at $z = 0$ where the antenna has a small gap. Since the current must vanish at the two ends of the antenna, the standing wave has form

$$I(z) \;=\; \hat{I} \times \sin\left(\frac{kL}{2} - k|z|\right) \tag{13}$$

where the amplitude $\hat{I}$ is related to the feed current I_0 as

$$\hat{I} \;=\; \frac{I_0}{\sin(kL/2)}\,. \tag{14}$$

Here is the plot for $L = \frac{5}{4}\lambda$:



For this current profile $I(z)$, the integral (11) becomes

$$\begin{aligned}
\mathbf{f}(\mathbf{n}) &= \frac{\mathbf{n}_3}{4\pi} \int\limits_{-L/2}^{+L/2} dz\, \hat{I} \sin\left(\tfrac{1}{2}kL \,-\, k|z|\right) \times \exp(-ik\cos\theta \times z) \\
&= \frac{\mathbf{n}_3}{4\pi} \int\limits_{0}^{+L/2} dz\, \hat{I} \sin\left(\tfrac{1}{2}kL \,-\, kz\right) \times \Big(\exp(-ik\cos\theta \times z) \,+\, \exp(+ik\cos\theta \times z)\Big)
\end{aligned} \tag{15}$$

where

$$\exp(-ik\cos\theta \times z) \,+\, \exp(+ik\cos\theta \times z) \;=\; 2\cos(k\cos\theta \times z) \tag{16}$$

and

$$\begin{aligned}
\sin\left(\tfrac{1}{2}kL \,-\, kz\right) \times 2\cos(k\cos\theta \times z) \;&=\; \sin\left(\tfrac{1}{2}kL \,-\, kz \,+\, k\cos\theta \times z\right) \\
&\quad +\; \sin\left(\tfrac{1}{2}kL \,-\, kz \,-\, k\cos\theta \times z\right).
\end{aligned} \tag{17}$$

Plugging these formulae back into the integral (15), we obtain

$$\begin{aligned}
\mathbf{f}(\mathbf{n}) &= \frac{\hat{I}\mathbf{n}_3}{4\pi} \int\limits_{0}^{+L/2} dz \Big(\sin\big(\tfrac{1}{2}kL \,-\, k(1-\cos\theta) \times z\big) \,+\, \sin\big(\tfrac{1}{2}kL \,-\, k(1+\cos\theta) \times z\big)\Big) \\
&= \frac{\hat{I}\mathbf{n}_3}{4\pi} \left[\frac{\cos\big(\tfrac{1}{2}kL \,-\, k(1-\cos\theta) \times z\big)}{k(1-\cos\theta)} \,+\, \frac{\cos\big(\tfrac{1}{2}kL \,-\, k(1+\cos\theta) \times z\big)}{k(1+\cos\theta)}\right]\Bigg|_{z=0}^{z=+L/2}
\end{aligned} \tag{18}$$

where

$$\text{at } z = +\tfrac{1}{2}L, \quad \cos\big(\tfrac{1}{2}kL - k(1\mp\cos\theta)\times z\big) \;=\; \cos\big(\tfrac{1}{2}kL - \tfrac{1}{2}kL \pm \tfrac{1}{2}kL\cos\theta\big) \;=\; \cos\big(\tfrac{1}{2}kL\cos\theta\big) \tag{19}$$

while

$$\text{at } z = 0, \quad \cos\big(\tfrac{1}{2}kL \,-\, k(1\mp\cos\theta) \times z\big) \;=\; \cos\big(\tfrac{1}{2}kL\big). \tag{20}$$

Consequently, on the last line of eq. (18),

$$
\begin{aligned}
\Big[\cdots\Big]\Big|_{z=0}^{z=+L/2} &= \cos\!\left(\tfrac{1}{2}kL\cos\theta\right)\times\left(\frac{1}{k(1-\cos\theta)}+\frac{1}{k(1+\cos\theta)}\right)\\
&\quad-\cos\!\left(\tfrac{1}{2}kL\right)\times\left(\frac{1}{k(1-\cos\theta)}+\frac{1}{k(1+\cos\theta)}\right)\\
&= \Big(\cos\!\left(\tfrac{1}{2}kL\cos\theta\right)-\cos\!\left(\tfrac{1}{2}kL\right)\Big)\times\frac{2}{k(1-\cos^2\theta)}\,,
\end{aligned}
\tag{21}
$$

and therefore

$$
\mathbf{f}(\mathbf{n}) = \frac{\hat{I}\mathbf{n}_3}{2\pi k\sin^2\theta}\Big(\cos\!\left(\tfrac{1}{2}kL\cos\theta\right)-\cos\!\left(\tfrac{1}{2}kL\right)\Big). \tag{22}
$$

Note that the direction of this vector amplitude $\mathbf{f}$ is $\mathbf{n}_3$, the direction of the antenna, so that

$$
\big\|\mathbf{n}_x\times\mathbf{f}(\mathbf{n}_x)\big\| = \|\mathbf{f}(\mathbf{n}_x)\|\times\sin\theta. \tag{23}
$$

Consequently, plugging the amplitude (22) into eq. (9) for the EM power radiated in the direction $\mathbf{n}_x=(\theta,\phi)$, we arrive at

$$
\frac{dP}{d\Omega} = \frac{Z_0|\hat{I}|^2}{8\pi^2}\times\left(\frac{\cos\!\left(\tfrac{1}{2}kL\times\cos\theta\right)-\cos\!\left(\tfrac{1}{2}kL\right)}{\sin\theta}\right)^2, \tag{24}
$$

or in terms of the feed current I_0 rather than the standing wave amplitude $\hat{I}$,

$$
\frac{dP}{d\Omega} = \frac{Z_0I_0^2}{8\pi^2\sin^2\!\left(\tfrac{1}{2}kL\right)}\times\left(\frac{\cos\!\left(\tfrac{1}{2}kL\times\cos\theta\right)-\cos\!\left(\tfrac{1}{2}kL\right)}{\sin\theta}\right)^2. \tag{25}
$$

The second factor in this formula gives us the angular dependence of the power radiated by the linear antenna. Let's consider a few common examples in some detail.

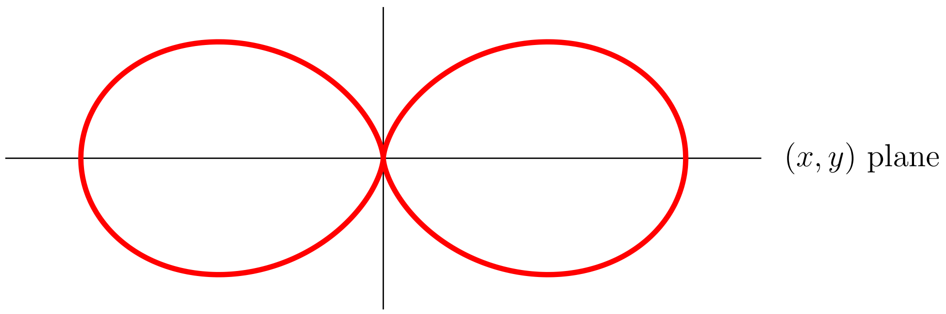
- A short dipole antenna, $L \ll \lambda$. For such a short dipole

$$\cos\left(\tfrac{1}{2}kL \times \cos\theta\right) \;-\; \cos\left(\tfrac{1}{2}kL\right) \;\approx\; \frac{(kL)^2}{8} \times (1 - \cos^2\theta) \tag{26}$$

and therefore

$$\frac{dP}{d\Omega} \;\propto\; \left(\frac{1 - \cos^2\theta}{\sin\theta}\right)^2 \;=\; \sin^2\theta. \tag{27}$$

Graphically, this direction-dependence of the radiated power is depicted on the *radiation pattern diagram*[⋆]



For the short dipole antenna, this radiation pattern diagram has a single main lobe centered at the horizontal plane, but this main lobe is rather thick, so a lot of radiation goes in directions well above or well belows the horizontal.

- A half-wavelength antenna, $L = \frac{1}{2}\lambda$, thus $\frac{1}{2}kL = \pi/2$. This is a very common antenna length for the UHF radio waves or microwaves. For such an antenna

$$\cos\left(\tfrac{1}{2}kL \times \cos\theta\right) \;-\; \cos\left(\tfrac{1}{2}kL\right) \;=\; \cos\left(\tfrac{\pi}{2} \times \cos\theta\right) \;-\; 0 \tag{28}$$

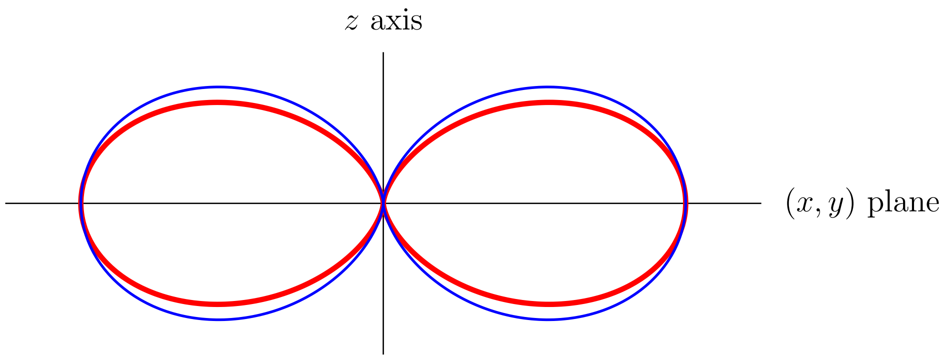
and hence

$$\frac{dP}{d\Omega} \;\propto\; \frac{\cos^2\left(\frac{\pi}{2} \times \cos\theta\right)}{\sin^2\theta}\,. \tag{29}$$

The radiation pattern diagram for the half-wavelength antenna is the thick red line on

⋆ For a general antenna, such a diagram should be three-dimensional, but thanks to the axial symmetry of the linear antenna, a 2D diagram like the above is sufficient.

this plot:



The thin blue line here is for comparison purposes: it's the radiation pattern diagram for a short-dipole antenna. We see that both radiation patterns have a single main lobe centered in the horizontal plane, bot for the half-dipole antenna this main lobe is not quite as thick as for the short-dipole antenna.

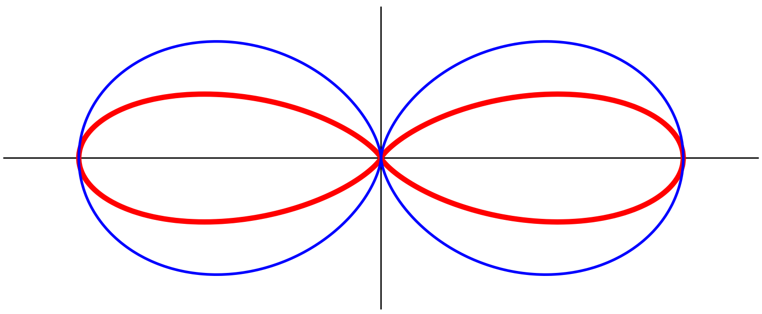
- Full wavelength antenna, $L = \lambda$ and hence $\frac{1}{2}kL = \pi$. This is also a common antenna length for the UHF radio waves and microwaves. For the full-wavelength antenna

$$\cos\left(\tfrac{1}{2}kL \times \cos\theta\right) \;-\; \cos\left(\tfrac{1}{2}kL\right) \;=\; \cos(\pi \times \cos\theta) \;+\; 1 \;=\; 2\cos^2\left(\tfrac{\pi}{2} \times \cos\theta\right), \qquad (30)$$

so the direction-dependence of the fill-wavelength antenna's radiation is

$$\frac{dP}{d\Omega} \;\propto\; \frac{\cos^4\left(\frac{\pi}{2} \times \cos\theta\right)}{\sin^2\theta}\,. \qquad (31)$$

The radiation patter diagram for the full wavelength antenna is



Again, we have a single main lobe centered in the horizontal plane, but this time it's quite thinner than for the short-dipole antenna.

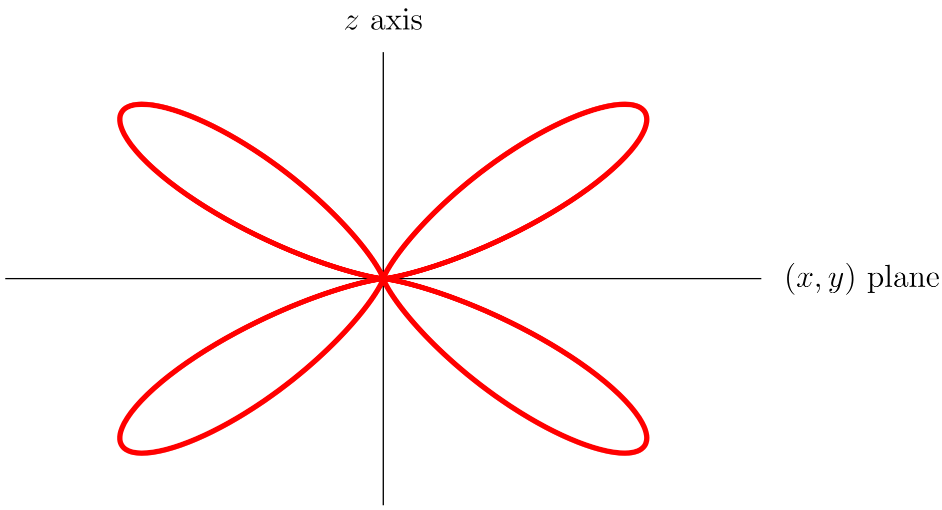
- Double wavelength antenna, $L = 2\lambda$, thus $\frac{1}{2}kL = 2\pi$. For this antenna

$$\begin{aligned} \cos\left(\tfrac{1}{2}kL \times \cos\theta\right) - \cos\left(\tfrac{1}{2}kL\right) &= \cos(2\pi \times \cos\theta) - 1 \\ &= -2\sin^2(\pi \times \cos\theta) \\ &= -8\sin^2\left(\tfrac{\pi}{2}\cos\theta\right)\cos^2\left(\tfrac{\pi}{2}\cos\theta\right) \end{aligned} \tag{32}$$

and therefore

$$\frac{dP}{d\Omega} \propto \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)\sin^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}. \tag{33}$$

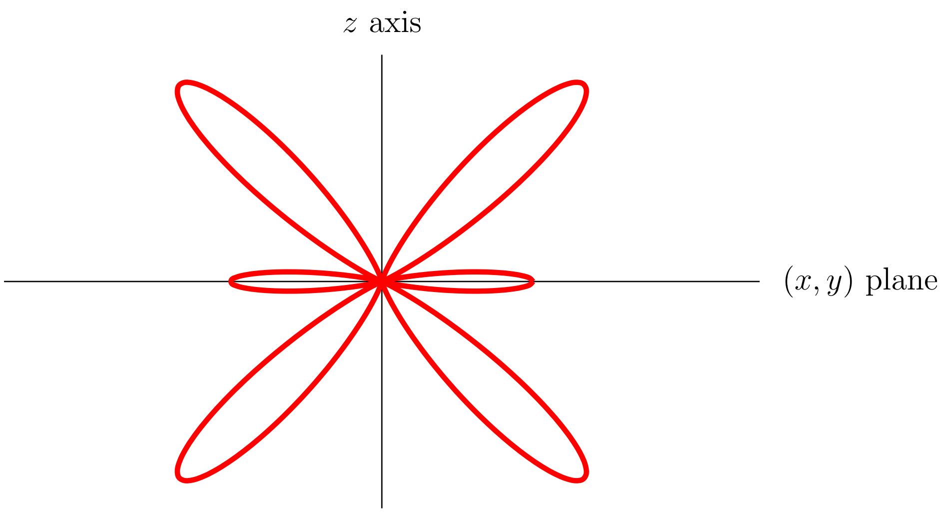
The radiation pattern diagram for the double-wavelength antenna is



Surprisingly, there are 2 major lobes, one centered at $\theta = 58°$ and the other at $\theta = 122°$ — *i.e.*, $32°$ above or below the horizontal, — while no radiation is emitted in the horizontal or vertical direction.

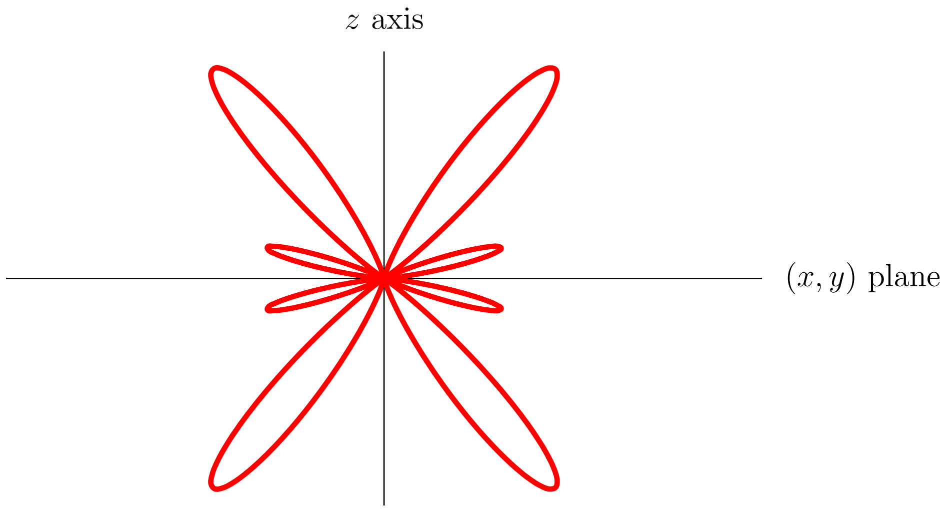
- For the longer antennas, $L = n\lambda$ for $n = 3, 4, 6, 10$, let me skip the formulae and simply plot the radiation pattern diagrams.

  * $L = 3\lambda$:



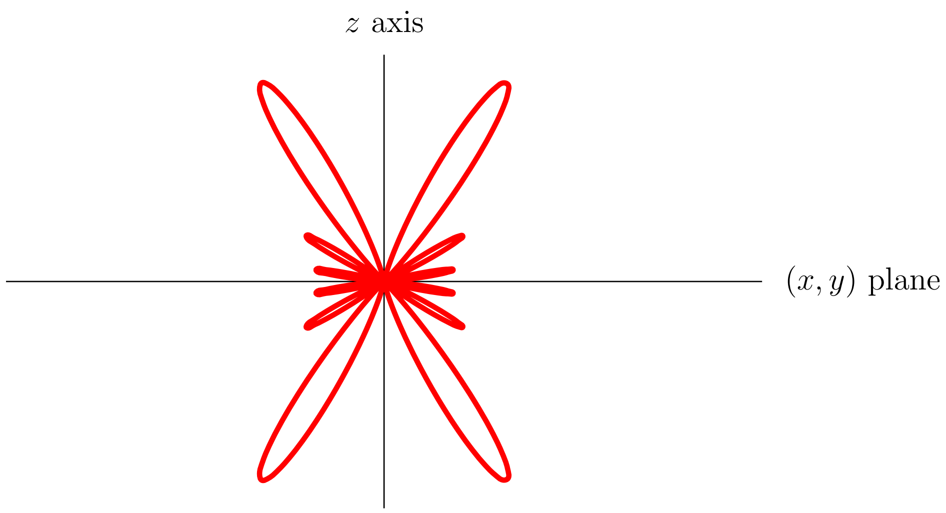
Note 2 diagonal major lobes and a minor horizontal lobe.

* $L = 4\lambda$:

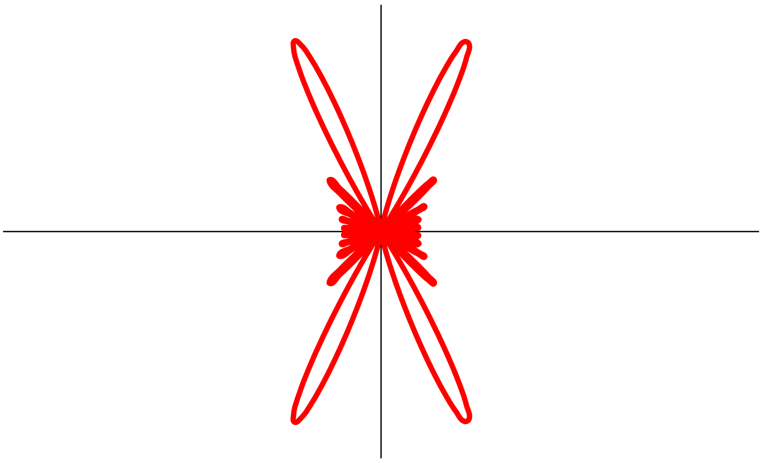


This time, we have two major lobes at steep angles from the horizontal, plus 2 minor lobes at shallower angles.

* $L = 6\lambda$:



* $L = 10\lambda$:



⋆ In general, the radiation pattern diagram for an antenna of length $L = n\lambda$ for some integer n has n lobes centered at angles θ which obtain from the transcendental equation

$$\begin{aligned} \pi n \tan\bigl(n\tfrac{\pi}{2}\,\cos\theta\bigr) &= -\frac{\cos\theta}{\sin^2\theta} \quad \text{for even } n, \\ \pi n \,\mathrm{ctan}\bigl(n\tfrac{\pi}{2}\,\cos\theta\bigr) &= +\frac{\cos\theta}{\sin^2\theta} \quad \text{for odd } n. \end{aligned} \tag{34}$$

Among the n lobes, the two major lobes are much larger than the other $N - 2$ minor lobes. In terms of their central θ angles, the major lobes corresponds to the largest and the smallest solutions of eqs. (34). In other words, the two major lobes are closest to the vertical direction of the antenna itself.

⋆ ⋆ ⋆

Now let's turn our attention from the directions in which the EM power is radiated by a long linear antenna to the net power radiated by it,

$$P_{\rm net} = \int\limits_{4\pi} d^2\Omega\,\frac{dP}{d\Omega} = \frac{Z_0 I_0^2}{4\pi^2\sin^2(kL/2)} \times \int\limits_{4\pi} d^2\Omega \left(\frac{\cos\bigl(\frac12 kL\times\cos\theta\bigr) - \cos\bigl(\frac12 kL\bigr)}{\sin\theta}\right)^2 . \quad (35)$$

Formally, the integral here evaluates to

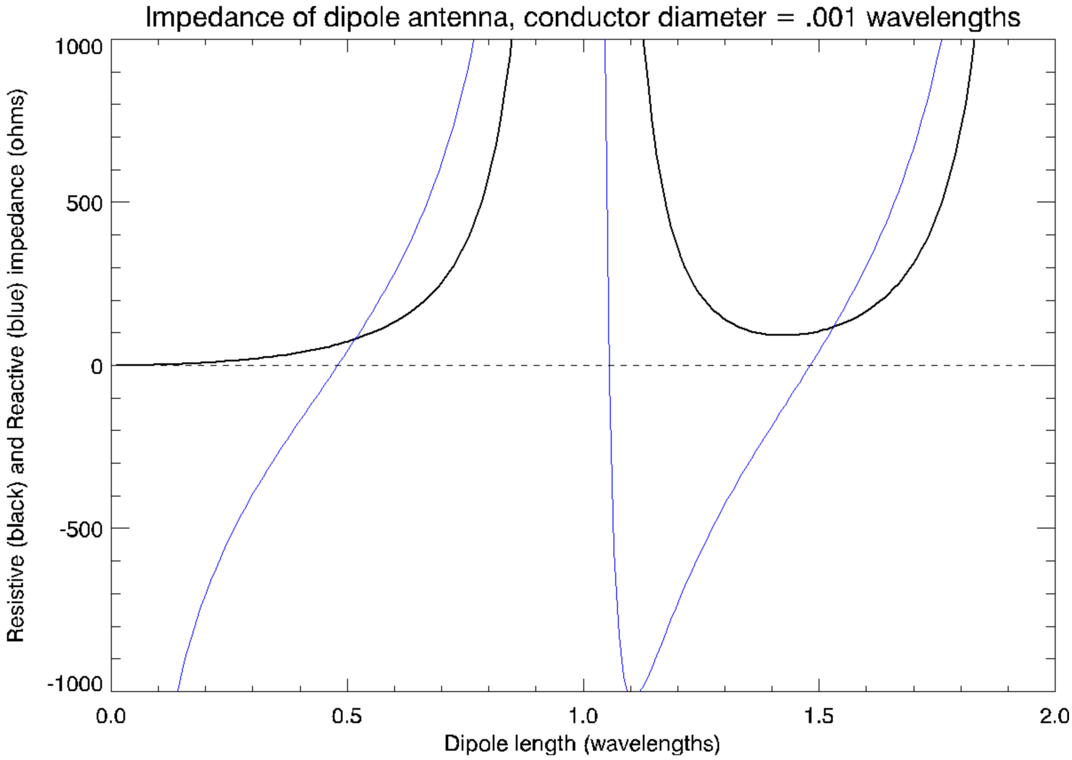
$$P_{\rm net} = \frac{Z_0 I_0^2}{4\pi\sin^2(kL/2)} \times \begin{bmatrix} 4\cos^2(kL/2)\times{\rm Cin}(kL) \\ -\cos(kL)\times{\rm Cin}(2kL) \\ -2\sin(kL)\times{\rm Si}(kL) \\ +\sin(kL)\times{\rm Si}(2kL) \end{bmatrix} . \quad (36)$$

where

$${\rm Si}(x) \stackrel{\rm def}{=} \int\limits_0^x \frac{\sin(t)}{t}\,dt \quad \text{and} \quad {\rm Cin}(x) \stackrel{\rm def}{=} \int\limits_0^x \frac{1-\cos(t)}{t}\,dt \quad (37)$$

are the sine and cosine integral function. You may look them up at this Wikipedia page, and if you are interested, read the references cited there. Alas, the class time is too short to spend it on studying these functions, so there is not much we can do with eq. (36) as it is.

Alternatively, I could have calculated the integral (35) numerically, but I doubt a Fortran code would be any more meaningful to you than eq. (36). Instead, let me simply copy the plot from the Wikipedia page on dipole antennas:



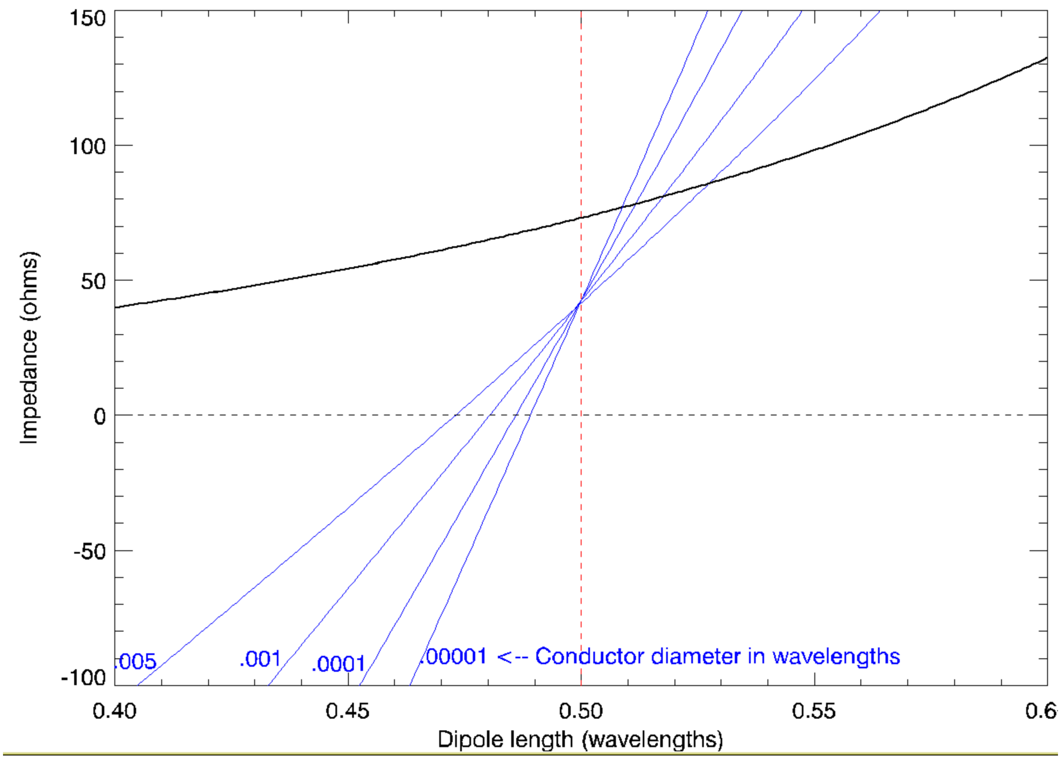
(38)

The black line on this plot shows the input resistance $R_{\rm in}$ — *i.e.*, the real part of the input impedance $Z_{\rm in}$ — of a center-fed linear antenna as a function of L/λ. The net power radiated by the antenna is related to this resistance as

$$P_{\rm net} \;=\; \frac{I_0^2}{2}\times R_{\rm in}\,. \tag{39}$$

Note very large input resistance for $L\approx\lambda$ and $L\approx 2\lambda$, or more generally whenever L/λ is close to an integer. This happens because for $L=\lambda\times$ an integer, the standing current wave in the linear antenna has a node at the center, so for a center-fed antenna the input current I_0 becomes very small compared to the standing wave amplitude $\hat{I}$. Consequently, we get a finite radiated power for a very small input current, which translates into a very large input resistance I_0.

The blue line on the plot (38) shows the input reactance $X_{\rm in}$. *i.e.*, the imaginary part of the input impedance $Z_{\rm in}$. The reactance is unrelated to the radiated power, so it cannot be obtained from our analysis in these notes. Instead, one has to solve the boundary conditions of the EM fields just outside the antenna, extract the voltage between the two halves of the antenna from the solution, and then relate it to the input current I_0 to get the complex input impedance $Z_{\text{in}} = V/I_0$. The reactance $\text{Im}(Z_{\text{in}})$ happens to depend logarithmically on the antenna's diameter a, so even for a very small diameter there is a notable difference between antennas with $a = 10^{-3}\lambda$ and $a = 10^{-4}\lambda$, *etc.*, *etc.* Here is a plot of reactances as functions of L/λ (in the neighborhood of $(L/\lambda) \approx 0.5$) for different antenna diameters:



(40)

Curiously, for antenna of length exactly $\frac{1}{2}\lambda$, the reactance seems to be diameter-independent

$$Z_{\text{in}} = (73 + j42.5)\,\Omega. \tag{41}$$

Antenna as a Boundary Problem

Earlier in these notes I have assumed that the current in the antenna propagates at the speed of light and therefore forms a 1D standing wave. Now let's try a more accurate analysis of the antenna current $I(z)$. For simplicity, let's assume that (1) the antenna is made from a round wire of uniform diameter a much smaller than the antenna's length or the wavelength; (2) the antenna wire is a perfect conductor, $\sigma \to \infty$; (3) the antenna is surrounded by the vacuum.

In light of the third assumption, everywhere outside the antenna

$$\mathbf{E} = \frac{ic^2}{\omega} \nabla \times \mathbf{B} = \frac{ic^2}{\omega} \nabla \times \nabla \times \mathbf{A} = \frac{ic^2}{\omega} \big(\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}\big). \tag{42}$$

Moreover, since the current in the antenna flows only in the $\pm z$ direction, the vector potential $\mathbf{A}(\mathbf{x})$ also points in the $\pm z$ direction, $\mathbf{A}(\mathbf{x}) = A_z(\mathbf{x})\mathbf{n}_3$, hence

$$\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} \tag{43}$$

and therefore

$$E_z(\mathbf{x}) = \frac{ic^2}{\omega} \left(\frac{\partial^2}{\partial z^2} - \nabla^2\right) A_z(\mathbf{x}). \tag{44}$$

Furthermore, the vector potential obeys the wave equation $\nabla^2 A_z = -k^2 A_z$, so eq. (44) for the vertical electric field becomes

$$E_z(\mathbf{x}) = \frac{ic^2}{\omega} \left(\frac{\partial^2}{\partial z^2} + k^2\right) A_z(\mathbf{x}). \tag{45}$$

Eq. (45) should hold everywhere outside the antenna, and in particular right next to the antenna's surface. But for a perfectly conducting antenna, the electric field immediately outside its surface should be normal to the surface, thus in cylindrical coordinates (s, ϕ, z)

$$E_z(s = a, z) = 0. \tag{46}$$

Or rather,

$$E_z(s = a, z) = V\delta(z) \tag{47}$$

where V is the voltage amplitude between the two halves of the antenna. Comparing eqs. (47)

and (45), we obtain an equation for the vector potential immediately outside the antenna:

$$\text{at } s=a \text{ and } -\tfrac{1}{2}L \le z \le +\tfrac{1}{2}L, \quad \left(\frac{\partial^2}{\partial z^2} + k^2\right) A_z(s,z) \;=\; \frac{-i\omega V}{c^2}\,\delta(z). \tag{48}$$

Solving this 1D differential equation gives us

$$A_z(s=a,z) \;=\; \alpha\cos(kz) \;+\; \beta\sin(kz) \;-\; \frac{iV}{2c}\,\sin(k|z|) \tag{49}$$

for some constant coefficients α and β. In light of the upside-down symmetry of the center-fed antenna, the β coefficient should vanish, but the α coefficient remains unknown at this stage, thus

$$\text{for } |z| \le \tfrac{1}{2}L, \quad A_z((s=a,z) \;=\; \alpha\cos(kz) \;-\; \frac{iV}{2c}\,\sin(k|z|). \tag{50}$$

At the same time, everywhere outside the antenna — and in particularly right outside the antenna's surface —

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\,\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,\mathbf{J}(\mathbf{y}). \tag{51}$$

In a perfectly conducting antenna, the current flows only on the surface due to the skin effect, thus

$$\mathbf{J}(s',\phi',z') \;=\; \frac{I(z')}{2\pi a}\,\delta(s-a)\,\mathbf{n}_3\,. \tag{52}$$

Plugging this current into eq. (51), we find that the vector potential immediately outside the antenna's surface is

$$A_z(s=a,\phi,z) \;=\; \frac{\mu_0}{8\pi^2}\int\limits_{-L/2}^{+L/2} dz'\,I(z')\int\limits_0^{2\pi} d\phi'\,\frac{\exp(ikR(z-z',\phi-\phi'))}{R(z-z',\phi-\phi')} \tag{53}$$

where $R(z-z',\phi-\phi')$ is the distance between the two points $(s=a,\phi,z)$ and $(s=a,\phi',z')$

on the antenna's surface,

$$R(z-z',\phi-\phi') \;=\; \sqrt{(z-z')^2\,+\,2a^2(1-\cos(\phi-\phi'))}\,. \tag{54}$$

In the context of the integral (53), let's define the *kernel*

$$K(z-z') \;=\int\limits_0^{2\pi}\frac{d\phi}{2\pi}\,\frac{\exp\left(ik\sqrt{(z-z')^2+2a^2(1-\cos\phi)}\right)}{\sqrt{(z-z')^2+2a^2(1-\cos\phi)}}\,. \tag{55}$$

In terms this kernel, the vector potential right at the antenna's surface is

$$A_z(s=a,z) \;=\; \frac{\mu_0}{4\pi}\int\limits_{-L/2}^{+L/2} dz'\,I(z')\times K(z-z')\,. \tag{56}$$

At the same time, the same vector potential at the antenna's surface is given by eq. (50), which gives us *an integral equation for the current $I(z)$*:

$$\text{for } |z|\le\tfrac{1}{2}L,\quad \frac{\mu_0}{4\pi}\int\limits_{-L/2}^{+L/2} dx'\,I(z')\times K(z-z') \;=\; \alpha\cos(kz)\;-\;\frac{iV}{2c}\,\sin(k|z|) \tag{57}$$

for some constant α.

Solving this integral equation is a major pain in the $***$, so let's no go there in this class. Instead, let me simply state that in the limit of an extremely thin antenna, the solution becomes the standing wave

$$I(z) \;\approx\; \hat{I}\sin\left(\tfrac{1}{2}kL\,-\,k|z|\right). \tag{58}$$

However, the relative corrections to this formula scale like $1/\ln(\lambda/a)$ rather than any positive power of the radius a, so even for $a\sim 10^{-4}\lambda$, the standing wave approximation could be off by 10–15%.