Charge Conjugation Symmetry

In the previous set of notes we followed Dirac's original construction of positrons as holes in the electron's Dirac sea. But the modern point of view is rather different: The Dirac sea is experimentally undetectable — it's simply one of the aspects of the physical vacuum state — and the electrons and the positrons are simply two related particle species.[⋆] Moreover, the electrons and the positrons have exactly the same mass but opposite electric charges.

Many other particle species exist in similar particle-antiparticle pairs. The particle and the corresponding antiparticle have exactly the same mass but opposite electric charges, as well as other conserved charges such as the lepton number or the baryon number. Moreover, the strong and the electromagnetic interactions — but not the weak interactions — respect the **change conjugation symmetry** which turns particles into antiparticles and vice verse,

$$\hat{\mathbf{C}}\,|\text{particle}(\mathbf{p},s)\rangle\;=\;|\text{antiparticle}(\mathbf{p},s)\rangle\,,\quad \hat{\mathbf{C}}\,|\text{antiparticle}(\mathbf{p},s)\rangle\;=\;|\text{particle}(\mathbf{p},s)\rangle\,,\tag{1}$$

for example $\hat{\mathbf{C}}\left|e^-(\mathbf{p},s)\right\rangle=\left|e^+(\mathbf{p},s)\right\rangle$ and $\hat{\mathbf{C}}\left|e^+(\mathbf{p},s)\right\rangle=\left|e^-(\mathbf{p},s)\right\rangle$. In light of this symmetry, deciding which particle species is particle and which is antiparticle is a matter of convention. For example, we know that the charged pions π^+ and π^- are each other's antiparticles, but it's up to our choice whether we call the π^+ mesons particles and the π^- mesons antiparticles or the other way around.

In the Hilbert space of the quantum field theory, the charge conjugation operator $\hat{\mathbf{C}}$ is a unitary operator which squares to 1, thus

$$\hat{\mathbf{C}}^2\;=\;1\quad\Longrightarrow\quad\hat{\mathbf{C}}^\dagger\;=\;\hat{\mathbf{C}}^{-1}\;=\;\hat{\mathbf{C}}.,\tag{2}$$

⋆ In condensed matter — say, in a piece of semiconductor — we may detect the filled electron states by making them interact with the outside world. But from the inside of the semiconductor, the electrons in the Fermi sea are simply features of the ground state, while the extra 'free' electrons or the holes are two species of quasiparticles which may appear in the excited states. Likewise, as long as we live and work inside the physical vacuum, the Dirac sea is simply an aspect of that physical vacuum, and we cannot detect it experimentally short of stepping outside our physical vacuum. And inside the physical vacuum, the electrons and the positrons are two related species of relativistic particles.

and it acts on the creation and annihilation operators of the charged particles according to

$$\hat{\mathbf{C}}\hat{a}^\dagger_{\mathbf{p},s}\hat{\mathbf{C}} \;=\; \hat{b}^\dagger_{\mathbf{p},s}\,, \quad \hat{\mathbf{C}}\hat{b}^\dagger_{\mathbf{p},s}\hat{\mathbf{C}} \;=\; \hat{a}^\dagger_{\mathbf{p},s}\,, \quad \hat{\mathbf{C}}\hat{a}_{\mathbf{p},s}\hat{\mathbf{C}} \;=\; \hat{b}_{\mathbf{p},s}\,, \quad \hat{\mathbf{C}}\hat{b}_{\mathbf{p},s}\hat{\mathbf{C}} \;=\; \hat{a}_{\mathbf{p},s}\,. \tag{3}$$

Note: these formulae apply to any charged particle species, be they bosons or fermions, of any spin.

Now consider the action of the charge conjugation on the quantum fields. For example, consider a complex scalar field $\Phi(x)$ for some spinless particles and antiparticles. Expanding $\hat{\Phi}(x)$ into annihilation and creation operators,

$$\hat{\Phi}(x) \;=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\left(e^{-ipx}\,\hat{a}_{\mathbf{p}} \,+\, e^{+ipx}\,\hat{b}^\dagger_{\mathbf{p}}\right)^{p^0=+E_bp} \tag{4}$$

and applying the $\hat{\mathbf{C}}$ operator according to eqs. (3), we have

$$\begin{aligned}
\hat{\mathbf{C}}\hat{\Phi}(x)\hat{\mathbf{C}} \;&=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\left(e^{-ipx}\times\hat{\mathbf{C}}\hat{a}_{\mathbf{p}}\hat{\mathbf{C}} \,+\, e^{+ips}\times\hat{\mathbf{C}}\hat{b}^\dagger_{\mathbf{p}}\hat{\mathbf{C}}\right)^{p^0=+E_{\mathbf{p}}} \\
&=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\left(e^{-ipx}\times\hat{b}_{\mathbf{p}} \,+\, e^{+ipx}\times\hat{a}^\dagger_{\mathbf{p}}\right)^{p^0=+E_{\mathbf{p}}} \\
&=\; \hat{\Phi}^\dagger(x),
\end{aligned} \tag{5}$$

and likewise,

$$\hat{\mathbf{C}}\hat{\Phi}^\dagger(x)\hat{\mathbf{C}} \;=\; \hat{\Phi}(x). \tag{6}$$

Consequently, *in the classical limit, the charge conjugation symmetry acts on scalar fields by complex conjugation,*

$$\mathbf{C}:\; \Phi(x) \;\leftrightarrow\; \Phi^*(x). \tag{7}$$

For the Dirac spinor fields, the action of the charge conjugation symmetry is a bit more involved. In the Weyl convention for the γ^μ matrices,

$$\hat{\mathbf{C}}\widehat{\Psi}(x)\hat{\mathbf{C}} \;=\; \gamma^2\widehat{\Psi}^*(x) \tag{8}$$

where the * superscript on a quantum field means Hermitian conjugation component by component but without transposing the column vector into a raw vector, thus

$$
\begin{aligned}
\text{for } \widehat{\Psi}(x) \ &= \ \begin{pmatrix} \hat{\psi}_1(x) \\ \hat{\psi}_2(x) \\ \hat{\psi}_3(x) \\ \hat{\psi}_4(x) \end{pmatrix}, \\
\widehat{\Psi}^\dagger(x) \ &= \ \left(\hat{\psi}_1^\dagger(x) \quad \hat{\psi}_2^\dagger(x) \quad \hat{\psi}_3^\dagger(x) \quad \hat{\psi}_4^\dagger(x) \right) \quad \text{while} \quad \widehat{\Psi}^*(x) \ = \ \begin{pmatrix} \hat{\psi}_1^\dagger(x) \\ \hat{\psi}_2^\dagger(x) \\ \hat{\psi}_3^\dagger(x) \\ \hat{\psi}_4^\dagger(x) \end{pmatrix}.
\end{aligned}
\tag{9}
$$

To see how this works, let's expand the fermionic fields into annihilation and creation operators:

$$
\widehat{\Psi}(x) \ = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \sum_s \Big(e^{-ipx} u(\mathbf{p},s) \times \hat{a}_{\mathbf{p},s} \ + \ e^{+ipx} v(\mathbf{p},s) \times \hat{b}^\dagger_{\mathbf{p},s} \Big)^{p^0=+E_\mathbf{p}}, \tag{10}
$$

$$
\widehat{\Psi}^\dagger(x) \ = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \sum_s \Big(e^{-ipx} v^\dagger(\mathbf{p},s) \times \hat{b}_{\mathbf{p},s} \ + \ e^{+ipx} u^\dagger(\mathbf{p},s) \times \hat{a}^\dagger_{\mathbf{p},s} \Big)^{p^0=+E_\mathbf{p}}, \tag{11}
$$

or equivalently

$$
\widehat{\Psi}^*(x) \ = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \sum_s \Big(e^{-ipx} v^*(\mathbf{p},s) \times \hat{b}_{\mathbf{p},s} \ + \ e^{+ipx} u^*(\mathbf{p},s) \times \hat{a}^\dagger_{\mathbf{p},s} \Big)^{p^0=+E_\mathbf{p}}. \tag{12}
$$

The $u(\mathbf{p},s)$ and $v(\mathbf{p},s)$ are the constant spinor coefficients of the plane-wave solutions of the Dirac equation,

$$
\begin{aligned}
(i\partial\!\!\!/ - m)\big(e^{-ipx} u(p,s)\big) \ &= \ 0 \quad \Longleftrightarrow \quad (p\!\!\!/ - m) u(p,s) \ = \ 0 \\
\text{and} \quad (i\partial\!\!\!/ - m)\big(e^{+ipx} v(p,s)\big) \ &= \ 0 \quad \Longleftrightarrow \quad (p\!\!\!/ + m) v(p,s) \ = \ 0,
\end{aligned}
\tag{13}
$$

see problem 4 of homework#7 for details. In particular, in that problem you should have seen that in the Weyl convention

$$
v(\mathbf{p},s) \ = \ \gamma^2 u^*(\mathbf{p},s) \quad \text{and} \quad u(\mathbf{p},s) \ = \ \gamma^2 v^*(\mathbf{p},s). \tag{14}
$$

Consequently, multiplying both sides of eq. (12) by the γ^2 matrix, we get

$$\begin{aligned}\gamma^2\widehat{\Psi}^*(x) &= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_\mathbf{p}}\sum_s\left(e^{-ipx}\gamma^2 v^*(\mathbf{p},s)\times\hat{b}_{\mathbf{p},s}\ +\ e^{+ipx}\gamma^2 u^*(\mathbf{p},s)\times\hat{a}^\dagger_{\mathbf{p},s}\right)^{p^0=+E_\mathbf{p}}\\ &= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_\mathbf{p}}\sum_s\left(e^{-ipx}u(\mathbf{p},s)\times\hat{b}_{\mathbf{p},s}\ +\ e^{+ipx}v(\mathbf{p},s)\times\hat{a}^\dagger_{\mathbf{p},s}\right)^{p^0=+E_\mathbf{p}}.\end{aligned} \tag{15}$$

At the same time, applying the charge conjugate symmetry (3) to each creation and annihilation operator in the expansion (10), we get

$$\begin{aligned}\hat{\mathbf{C}}\widehat{\Psi}(x)\hat{\mathbf{C}} &= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_\mathbf{p}}\sum_s\left(e^{-ipx}u(\mathbf{p},s)\times\hat{\mathbf{C}}\hat{a}_{\mathbf{p},s}\hat{\mathbf{C}}\ +\ e^{+ipx}v(\mathbf{p},s)\times\hat{\mathbf{C}}\hat{b}^\dagger_{\mathbf{p},s}\hat{\mathbf{C}}\right)^{p^0=+E_\mathbf{p}}\\ &= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_\mathbf{p}}\sum_s\left(e^{-ipx}u(\mathbf{p},s)\times\hat{b}_{\mathbf{p},s}\ +\ e^{+ipx}v(\mathbf{p},s)\times\hat{a}^\dagger_{\mathbf{p},s}\right)^{p^0=+E_\mathbf{p}},\\ &\quad\text{same as the bottom line of eq. (15),}\end{aligned} \tag{16}$$

hence

$$\hat{\mathbf{C}}\widehat{\Psi}(x)\hat{\mathbf{C}}\ =\ \gamma^2\widehat{\Psi}^*(x). \tag{8}$$

Note: the γ^2 matrix here is specific to the Weyl convention for the Dirac matrices. More generally, we have

$$\hat{\mathbf{C}}\widehat{\Psi}(x)\hat{\mathbf{C}}\ =\ C\widehat{\Psi}^*(x) \tag{17}$$

where C is a convention-dependent matrix acting on Dirac indices which obeys

$$\gamma^*_\mu C\ =\ -C\gamma_\mu\quad\text{and}\quad C^*C\ =\ 1. \tag{18}$$

In particular, in the Majorana convention where all 4 γ_μ matrices are imaginary, $C = 1$ and the charge conjugation acts by complex conjugation of the spinor field. But in all other conventions, this complex conjugation is accompanied by multiplying by the appropriate matrix C.

Neutral Particles

Particle species which have any kind of conserved quantum numbers — the electric charge, the baryon number, the lepton number, whatever — come in particle-antiparticle pairs, and for all such pairs the charge conjugation symmetry swaps particles with antiparticles,

$$\hat{\mathbf{C}}\,|\text{particle}(\mathbf{p},s)\rangle \;=\; |\text{antiparticle}(\mathbf{p},s)\rangle\,, \quad \hat{\mathbf{C}}\,|\text{antiparticle}(\mathbf{p},s)\rangle \;=\; |\text{particle}(\mathbf{p},s)\rangle\,, \tag{1}$$

But some particles — like the photons — are *inherently neutral:* they do not have any conserved charges, and the particles are identical to the antiparticles,

$$|\text{antiparticle}(\mathbf{p},s)\rangle \;=\; |\text{particle}(\mathbf{p},s)\rangle\,. \tag{19}$$

When the charge conjugation symmetry acts on such particles, it turns them into themselves up to an overall sign of the quantum state,

$$\hat{\mathbf{C}}\,|\text{particle}(\mathbf{p},s)\rangle \;=\; \pm\,|\text{same particle}(\mathbf{p},s)\rangle\,. \tag{20}$$

The species-dependent $\pm$ sign here is called the *C-parity* of the particle. For example, the photons are C-odd, $\hat{\mathbf{C}}\,|\gamma(p,\lambda)\rangle = -\,|\gamma(p,\lambda)\rangle$. Indeed, since the charge conjugation flips the sign of the electric charge, the electric current $J^\mu(x)$ must be C-odd,

$$\hat{\mathbf{C}}\hat{J}^\mu(x)\hat{\mathbf{C}} \;=\; -\hat{J}^\mu(x). \tag{21}$$

Hence, to keep the coupling $A^\mu J_\mu$ of the EM field A^μ to the electric current invariant under the charge conjugation, the EM potentials $A^\mu(x)$ must also be C-odd,

$$\hat{\mathbf{C}}\hat{A}^\mu(x)\hat{\mathbf{C}} \;=\; -\hat{A}^\mu(x). \tag{22}$$

Consequently, the EM tension fields $F^{\mu\nu}(x)$ are also C-odd, and when we expand them into photonic creation and annihilation operators, all such operators must be C-odd. In particle terms, this means $\hat{\mathbf{C}}\,|\gamma(p,\lambda)\rangle = -\,|\gamma(p,\lambda)\rangle$.

Other particles have different C-parities, and this affects their interactions and decays. For example, the neutral pion π^0 is C-even, $\hat{\mathbf{C}}\left|\pi^0(p)\right\rangle = +\left|\pi^0(p)\right\rangle$. Consequently, when it decays electromagnetically, the final state must also be C-even, for example an even number of photons, thus

$$\pi^0 \;\to\; \gamma+\gamma, \quad \pi^0 \to\; \gamma+\gamma+\gamma+\gamma, \quad \ldots. \tag{23}$$

OOH, the charge conjugation symmetry forbids π^0 to decays into C-odd states such as odd numbers of photons,

$$\pi^0 \;\not\to\; \gamma+\gamma+\gamma,\ldots. \tag{24}$$

Experimentally, 98.8% of neutral pion decays are into pairs of photons, $\pi^0 \to \gamma+\gamma$. The less common decay mode (branching ratio about 1.2%) is $\pi^0 \to \gamma + e^+ + e^-$, where the final state is C-even. There also rare decay modes into other C-even final states, but never into C-odd states such as $\pi^0 \to 3\gamma$.

Another neutral meson ρ^0 is C-odd, so it may decay electromagnetically into 3 photons but not into 2 photons,

$$\rho^0 \;\not\to\; 2\gamma \quad \text{but} \quad \rho^0 \;\to\; 3\gamma \text{ is OK.} \tag{25}$$

The strong interactions also respect the C-parity, so the neutral ρ meson does not decay into a pair of neutral pions,

$$\rho^0 \;\not\to\; \pi^0+\pi^0. \tag{26}$$

On the other hand, the ρ^0 can decay into a C-odd state of two charged pions,

$$\rho^0 \;\to\; \pi^+ + \pi^-, \tag{27}$$

which is the dominant decay mode of the ρ^0 meson.

Both π^0 and ρ^0 mesons are bound states of quark-antiquark pairs, $u+\bar{u}$ or $d+\bar{d}$. The spatial wave functions for both mesons have $L=0$ but the spin wave functions are different, so the π^0 mesons has $S = 0$ while the ρ^0 meson has $S = 1$. In your next homework#8 , you should see that the C-parity of a fermion-antifermion bound state is

$$C = (-1)^L \times (-1)^S, \tag{28}$$

and that's why π^0 is C-even while ρ^0 is C-odd.

Majorana Fermions

Experimentally, the particle data book lists many inherently neutral bosons of different spins, but all the known fermionic particles are charged: Even if they are electrically neutral like the neutron, they have a baryon number or a lepton number. But theoretically, the inherently neutral fermions may exist, and if we are so lucky they may be discovered tomorrow. For example, if the world is approximately supersymmetric then the photon has a spin $= \frac{1}{2}$ superpartner — called the *photino* — which is inherently neutral, just like the photon. Also, in many models of dark matter, it is made of heavyish spin $= \frac{1}{2}$ inherently neutral fermions.

The inherently neutral fermions are quanta of the *Majorana spinor fields,* which are spin $= \frac{1}{2}$ analogues of real — as opposed to complex — scalar fields for the inherently neutral scalar particle. Indeed, if a scalar field $\hat{\Phi}(x)$ has inherently neutral quanta, $\hat{\mathbf{C}}\,|S(p)\rangle = \pm\,|S(p)\rangle$, then

$$\hat{\mathbf{C}}\hat{\Phi}(x)\hat{\mathbf{C}} = \pm\hat{\Phi}(x) \quad\Longrightarrow\quad \hat{\Phi}^\dagger(x) = \hat{\mathbf{C}}\hat{\Phi}(x)\hat{\mathbf{C}} = \pm\hat{\Phi}(x), \tag{29}$$

so $\hat{\Phi}(x)$ must be either Hermitian or anti-Hermitian. In the classical limit, this means that either $\Phi(x)$ itself or $i \times \Phi(x)$ is a real field.

For an inherently neutral spin $= \frac{1}{2}$ fermion, $\hat{\mathbf{C}}\,|F(\mathbf{p}, s)\rangle = \pm\,|F(\mathbf{p}, s)\rangle$, the corresponding quantum spinor field $\widehat{\Psi}(x)$ obeys

$$\hat{\mathbf{C}}\widehat{\Psi}(x)\hat{\mathbf{C}} = \pm\widehat{\Psi}(x). \tag{30}$$

But we also have

$$\hat{\mathbf{C}}\widehat{\Psi}(x)\hat{\mathbf{C}} = \gamma^2\widehat{\Psi}^*(x) \tag{31}$$

(in the Weyl convention), hence

$$\gamma^2 \widehat{\Psi}^*(x) \;=\; \pm\widehat{\Psi}(x). \tag{32}$$

Or in other conventions for Dirac's matrices,

$$C\widehat{\Psi}^*(x) \;=\; \pm\widehat{\Psi}(x), \tag{33}$$

for example in the Majorana convention $C = 1$ hence

$$\widehat{\Psi}^*(x) \;=\; \pm\widehat{\Psi}(x). \tag{34}$$

A spinor field obeying such a Hermiticity condition — or a skewed Hermiticity condition (33) such as (32) — is called a Majorana spinor.

Let's count the degrees of freedom of the Majorana spinor field. There are 4 component fields $\Psi_\alpha(x)$, which are either real (in the Majorana convention) or linearly related to their complex conjugates — $\Psi^* = \pm C^{-1}\Psi$ — which makes them equivalent to 4 independent real fields. But the Dirac equation for these component fields is first-order in spacetime derivatives, so each real field carries only a $\frac{1}{2}$ degree of freedom. Consequently, the net number of degrees of freedom is

$$\#\text{DoF} \;=\; 4\times\tfrac{1}{2} \;=\; 2. \tag{35}$$

In particle terms, this means 2 distinct quantum states for each on-shell momentum p^μ. And indeed, the Majorana spinor fields correspond to inherently real particles of spin $= \frac{1}{2}$, hence a single particle species with 2 spin states, for the total of $1 \times 2 = 2$ degrees of freedom.

By comparison, a Dirac spinor field has 4 complex components $\Psi_\alpha(x)$ without any linear relations to their complex conjugates, so altogether they are equivalent to $4 \times 2 = 8$ independent real fields. Since the Dirac equation is first order in ∂_μ, each of these 8 fields carries $\frac{1}{2}$ of a degree of freedom, for the total of

$$\#\text{DoF} \;=\; 8\times\tfrac{1}{2} \;=\; 4, \tag{36}$$

twice as many as the Majorana spinor field.

In particle terms, the Dirac spinor field encodes two distinct particle species, the fermion and the antifermion, each having spin $= \frac{1}{2}$. Consequently, for each on-shell momentum p we may chose either of the two species, and for each species we have 2 spin states, thus $2 \times 2 = 4$ states altogether, in perfect agreement with the Dirac field having 4 degrees of freedom.

Weyl Spinor fields

For completeness sake, let's also consider the massless Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$. As you (should) have learned in homework#7 , problem 3, a massless Dirac spinor fields decomposes into two independent Weyl spinor fields: In the Weyl convention,

$$\Psi_{\text{Dirac}}(x) \;=\; \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}, \tag{37}$$

and

$$\mathcal{L}_D \;=\; \overline{\Psi}_D(i\gamma^\mu\partial_\mu)\Psi_D \;=\; i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L \;+\; i\psi_R\sigma^\mu\partial_\mu\psi_R\,. \tag{38}$$

For a massive Dirac field, this Lagrangian would have extra terms involving both ψ_L and ψ_R at the same time, but for $m = 0$ such terms disappear and the two Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$ become independent. If one wishes, one may even have a $\psi_L(x)$ without the $\psi_R(x)$ or vice verse.

Each of the Weyl spinors has two independent complex components, and there are no linear relations between these components and their complex conjugates. Consequently, each Weyl spinor is equivalent to $2 \times 2 = 4$ real fields. But the Lagrangian (38) is linear in the derivatives of the Weyl spinors, hence first-order (in ∂_μ) Weyl equations, which means each real component contributes $\frac{1}{2}$ a degree of freedom. Thus altogether, each Weyl spinor field — left handed or right-handed — has $4 \times \frac{1}{2} = 2$ degrees of freedom.

To see the particle content of these degrees of freedom, let's look at the plane-wave solutions of the Dirac equation for ultra-relativistic energies $E \gg m$ and definite helicities $\lambda = \pm\frac{1}{2}$. These solutions are spelled out in eq. (19) of homework#7 — and deriving them was your task. Anyway, in the $m \to 0$ limit and in the Weyl convention (37), these solutions become

$$
\begin{aligned}
u(p,\lambda=-\tfrac{1}{2}) &= \sqrt{2E}\begin{pmatrix}\xi_L\\0\end{pmatrix}, & u(p,\lambda=+\tfrac{1}{2}) &= \sqrt{2E}\begin{pmatrix}0\\\xi_R\end{pmatrix},\\
v(p,\lambda=-\tfrac{1}{2}) &= -\sqrt{2E}\begin{pmatrix}0\\\eta_L\end{pmatrix}, & v(p,\lambda=+\tfrac{1}{2}) &= \sqrt{2E}\begin{pmatrix}\eta_R\\0\end{pmatrix}.
\end{aligned}
\tag{39}
$$

In particular, the $u(\lambda=-\frac{1}{2})$ and the $v(\lambda=+\frac{1}{2})$ plane waves have only 2 upper components of the Dirac spinor but not the two lower components; in Weyl spinor terms, this means that the $u(\lambda=-\frac{1}{2})$ and the $v(\lambda=+\frac{1}{2})$ involve only the LH spinor ψ_L but not the RH spinor ψ_R. Similarly, the $u(\lambda=+\frac{1}{2})$ and $v(\lambda=-\frac{1}{2})$ plane waves have only 2 lower components but no upper components, so in terms of the Weyl spinors they involve the RH spinor ψ_R but not the LH Weyl spinor ψ_L. Consequently,

$$
\begin{aligned}
\hat{\psi}_L(x) &= \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\Big(e^{-ipx}\times u(p,\lambda=-\tfrac{1}{2})_{\text{upper}}\times\hat{a}(p,\lambda=-\tfrac{1}{2})\\
&\qquad\qquad +\ e^{+ipx}\times v(p,\lambda=+\tfrac{1}{2})_{\text{upper}}\times\hat{b}^\dagger(p,\lambda=+\tfrac{1}{2})\Big), && (40)\\
\hat{\psi}^\dagger_L(x) &= \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\Big(e^{-ipx}\times v^\dagger(p,\lambda=+\tfrac{1}{2})_{\text{upper}}\times\hat{b}(p,\lambda=+\tfrac{1}{2})\\
&\qquad\qquad +\ e^{+ipx}\times u^\dagger(p,\lambda=-\tfrac{1}{2})_{\text{upper}}\times\hat{a}^\dagger(p,\lambda=-\tfrac{1}{2})\Big), && (41)\\
\hat{\psi}_R(x) &= \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\Big(e^{-ipx}\times u(p,\lambda=+\tfrac{1}{2})_{\text{lower}}\times\hat{a}(p,\lambda=+\tfrac{1}{2})\\
&\qquad\qquad +\ e^{+ipx}\times v(p,\lambda=-\tfrac{1}{2})_{\text{lower}}\times\hat{b}^\dagger(p,\lambda=-\tfrac{1}{2})\Big), && (42)\\
\hat{\psi}^\dagger_R(x) &= \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\Big(e^{-ipx}\times v^\dagger(p,\lambda=-\tfrac{1}{2})_{\text{lower}}\times\hat{b}(p,\lambda=-\tfrac{1}{2})\\
&\qquad\qquad +\ e^{+ipx}\times u^\dagger(p,\lambda=+\tfrac{1}{2})_{\text{lower}}\times\hat{a}^\dagger(p,\lambda=+\tfrac{1}{2})\Big), && (43)
\end{aligned}
$$

which means that

- ⋆ Together, the left-handed Weyl spinor field $\hat{\psi}_L(x)$ and its Hermitian conjugate $\hat{\psi}^\dagger_L(x)$ create and annihilate the left-handed fermion and the right-handed antifermion.
- ⋆ Together, the right-handed Weyl spinor field $\hat{\psi}_R(x)$ and its Hermitian conjugate $\hat{\psi}^\dagger_R(x)$ create and annihilate the right-handed fermion and the left-handed antifermion.

- Note: for the fermions, the helicity of the particle matches the *chirality* of the Weyl spinor field: they are either both left-handed or both right-handed. But for the antifermions, the particle's helicity is opposite from the Weyl spinor field's chirality: The LH spinor field comes with the RH antifermion helicity, while the RH spinor field comes with the LH antifermion helicity.

⋆ In field theories which happen to have only the one Weyl spinor and its conjugate but not the other Weyl spinor of opposite chirality, the massless fermionic quanta have only one helicity state per species.

  ∗ Specifically, if we have only the LH Weyl spinor field $\hat{\psi}_L(x)$ but not the RH field $\hat{\psi}_R(x)$, then the fermions have helicity $\lambda = -\frac{1}{2}$ but never $\lambda = +\frac{1}{2}$, while the antifermions have $\lambda = +\frac{1}{2}$ but never $\lambda = -\frac{1}{2}$. Note: *This is exactly what happens to the massless neutrinos and antineutrinos in the Standard Model!*

  ∗ Likewise, if we have only the RH Weyl spinor $\hat{\psi}_R(x)$ but not the LH Weyl spinor $\hat{\psi}_L(x)$, then the fermions have helicity $\lambda = +\frac{1}{2}$ but never $\lambda = -\frac{1}{2}$, while the antifermions have $\lambda = -\frac{1}{2}$ but never $\lambda = +\frac{1}{2}$.

- Either way, the quanta of a single Weyl spinor field and its conjugate come in two species — the fermion and the antifermion, — but they have only one helicity state per species, hence $2 \times 1 = 2$ quantum states per on-shell momentum p. And that's how a single Weyl spinor field has two degrees of freedom.

Majorana–Weyl Equivalence

Thus far, we saw 3 kinds of spinor fields having 2 degrees of freedom: The Majorana spinor, the LH Weyl spinor, and the RH Weyl spinor. Actually, in 3+1 spacetime dimensions all these spinor fields are mathematically equivalent to each other,

$$\text{Majorana} \ \cong\ \text{LH Weyl} \ \cong\ \text{RH Weyl}. \tag{44}$$

Let's start with the two Weyl spinors. As you saw (or at least should have seen) in problem 3(a) of homework#7 , under the continuous Lorentz symmetries, the two Weyl spinor transform equivalently to the complex conjugates of each other:

$$\begin{aligned}
\text{for } x'^{\mu} &= L^{\mu}{}_{\nu} x^{\nu} : \\
\psi'_L(x') &= M_L(L)\psi_L(x), \quad \psi'_R(x') \;=\; M_R(L)\psi_R(x), \\
\text{while} \quad \sigma_2 \psi'^*_L(x') &= M_R(L) \times \sigma_2 \psi^*_L(x), \quad \sigma_2 \psi'^*_R(x') \;=\; M_L(L) \times \sigma_2 \psi^*_R(x).
\end{aligned} \tag{45}$$

Moreover, let's say we have only $\psi_L(x)$ (and its conjugate) but no $\psi_R(x)$, and let's define $\chi_R(x) = \sigma_2 \psi^*_L(x)$. From the Lorentz point of view, the $\chi_R(x)$ is a RH Weyl spinor, and its Lagrangian is also appropriate to a RH Weyl spinor. Indeed, if $\psi_L(x)$ has a massless Lagrangian

$$\mathcal{L} \;=\; i\psi^\dagger_L \bar{\sigma}^\mu \partial_\mu \psi_L, \tag{46}$$

then in terms of χ_R this Lagrangian becomes

$$\mathcal{L} \;=\; i\chi^\dagger_R \sigma^\mu \partial_\mu \chi_R. \tag{47}$$

Proof:

$$\begin{aligned}
\mathcal{L} &= i\big(\psi_L = \sigma_2 \chi^*_R\big)^\dagger \bar{\sigma}^\mu \partial_\mu \big(\psi_L = \sigma_2 \chi^*_R\big) \\
&= i\chi^\top_R \sigma^2 \bar{\sigma}^\mu \sigma_2 \partial_\mu \chi^*_R \\
&\qquad \langle\!\langle \text{ transposing, and getting an extra minus sign } \rangle\!\rangle \\
&\qquad \langle\!\langle \text{ due to anticommutation of } \chi_R \text{ and } \chi^*_R \rangle\!\rangle \\
&= -i\big(\partial_\mu \chi^\dagger_R\big)\big(\sigma_2 \bar{\sigma}_\mu \sigma_2\big)^\top \chi_R \\
&= -i\big(\partial_\mu \chi^\dagger_R\big)\sigma^\mu \chi_R \quad \langle\!\langle \text{ because } (\sigma_2 \bar{\sigma}^\mu \sigma_2)^\top = \sigma^\mu \rangle\!\rangle \\
&= +i\chi^\dagger_R \sigma^\mu \partial_\mu \chi_R \;-\; i\partial_\mu\big(\chi^\dagger_R \sigma^\mu \chi_R\big)
\end{aligned} \tag{48}$$

where the second term on the bottom line is a total derivative. Disregarding this terms since it does not contribute to the action $S = \int \mathcal{L} d^4x$, we end up with the correct free Lagrangian (47) for a mass less RH Weyl spinor field χ_R. Thus, a LH Weyl field $\psi_L(x)$ (together with its conjugate) is physically equivalent to the RH Weyl field $\chi_R(x)$ (together with its conjugate).

Likewise, a RH Weyl field $\psi_R(x)$ is physically equivalent to a LH Weyl field $\chi_L(x) = \sigma_2 \psi^*_R(x)$.

Now consider a Majorana spinor $\Psi_M(x)$ obeying $\gamma^2\Psi^*_M(x) = \pm\Psi_M(x)$. In terms of the LH and RH Weyl components of $\Psi_M(x)$, this condition becomes

$$\pm\begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix} = \begin{pmatrix}0 & +\sigma_2\\ -\sigma_2 & 0\end{pmatrix}\times\begin{pmatrix}\psi^*_L(x)\\ \psi^*_R(x)\end{pmatrix} = \begin{pmatrix}+\sigma_2\psi^*_R(x)\\ -\sigma_2\psi^*_L(x)\end{pmatrix}, \tag{49}$$

or in explicit Weyl spinor components

$$\psi_L(x) \;=\; \pm\sigma_2\psi^*_R(x) \quad\Longleftrightarrow\quad \psi_R(x) \;=\; \mp\sigma_2\psi^*_L(x). \tag{50}$$

In other words, the LH and the RH Weyl spinor components of a Majorana spinor field are equivalent to each other conjugates!

Therefore, if we take a LH Weyl spinor field $\psi_L(x)$, then

$$\Psi_M(x) \;=\; \begin{pmatrix}\psi_L(x)\\ -\sigma_2\psi^*_L(x)\end{pmatrix} \tag{51}$$

is a Majorana spinor, and its Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}\overline{\Psi}_M(i\gamma^\mu\partial_\mu - m)\Psi_M \tag{52}$$

becomes in terms of the ψ_L

$$\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}\overline{\Psi}_M\gamma^0(i\gamma^\mu\partial_\mu - m)\Psi_M\\
&= \frac{1}{2}\left(\,\psi^\dagger_L \quad -\psi^\top_L\sigma_2\,\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-m & i\sigma^\mu\partial_\mu\\ i\bar\sigma^\mu\partial_\mu & -m\end{pmatrix}\begin{pmatrix}\psi_L(x)\\ -\sigma_2\psi^*_L(x)\end{pmatrix}\\
&= \frac{1}{2}\left(\,\psi^\dagger_L \quad -\psi^\top_L\sigma_2\,\right)\begin{pmatrix}i\bar\sigma^\mu\partial_\mu & -m\\ -m & i\sigma^\mu\partial_\mu\end{pmatrix}\begin{pmatrix}\psi_L(x)\\ -\sigma_2\psi^*_L(x)\end{pmatrix}\\
&= \frac{i}{2}\,\psi^\dagger_L\bar\sigma^\mu\partial_\mu\psi_L \;+\; \frac{i}{2}\big(\sigma_2\psi^*_L\big)^\dagger\sigma^\mu\partial_\mu\big(\sigma_2\psi^*_L\big) \;+\; \frac{m}{2}\big(\sigma_2\psi^*_L\big)^\dagger\psi_L \;+\; \frac{m}{2}\psi^\dagger_L\big(\sigma_2\psi^*_L\big)\\
&\qquad \langle\!\langle\text{ similarly to eq. (48) }\rangle\!\rangle\\
&= i\psi^\dagger_L\bar\sigma^\mu\partial_\mu\psi_L \;+\; \text{a total derivative} \;+\; \frac{m}{2}\,\psi^\top_L\sigma_2\psi_L \;+\; \frac{m}{2}\,\psi^\dagger_L\sigma_2\psi^*_L.
\end{aligned} \tag{53}$$

Thus, a standalone Majorana spinor field is physically equivalent to a standalone massive LH Weyl spinor field plus its Hermitian conjugate, — but without any other fields. And in a similar way, the same standalone Majorana spinor is equivalent to a standalone massive RH Weyl spinor field plus its conjugate,

$$\begin{aligned}
\Psi_M(x) &= \begin{pmatrix} +\sigma_2\psi^*_R(x) \\ \psi_R(x) \end{pmatrix}, \\
\mathcal{L} &= \tfrac{1}{2}\overline{\Psi}_M\gamma^0(i\gamma^\mu\partial_\mu - m)\Psi_M \\
&= i\psi^\dagger_R\sigma^\mu\partial_\mu\psi_R \;+\; \text{a total derivative} \;+\; \frac{m}{2}\,\psi^\top_R\sigma_2\psi_R \;+\; \frac{m}{2}\,\psi^\dagger_R\sigma_2\psi^*_R.
\end{aligned} \tag{54}$$

The mass terms in eqs. (53) and (54) are called *Majorana mass terms.* Unlike the Dirac mass terms in the Dirac Lagrangian in the Weyl spinor language

$$\begin{aligned}
\Psi_D(x) &= \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}, \\
\mathcal{L} &= \overline{\Psi}_D\gamma^0(i\gamma^\mu\partial_\mu - m)\Psi_D \\
&= i\psi^\dagger_L\bar{\sigma}^\mu\partial_\mu\psi_L \;+\; i\psi^\dagger_R\sigma^\mu\partial_\mu\psi_R \;-\; m\psi^\dagger_L\psi_R \;-\; m\psi^\dagger_R\psi_L
\end{aligned} \tag{55}$$

which connect the two Weyl spinors to the conjugates of each other, the Majorana mass terms connect each spinor field $\psi^{\mathbf{p},s}_L$, $\psi^\dagger_L$, $\psi^{\mathbf{p},s}_R$, or $\psi^\dagger_R$ to to itself. Consequently, to accomodate the fermionic anticimmutativity of the spinor fields, the Majorana mass terms are antisymmetric in the spin indices, that's why they involve the σ_2 matrix.

Now let's focus on the LH Weyl spinor $\psi_L(x)$. For $m \neq 0$, the Majorana mass terms in the Lagrangian (53) break the global phase symmetry of the LH Weyl field, $\psi'_L(x) = e^{i\theta}\psi_L(x)$ Consequently, whatever charges the ψ_L's quanta might have, they are no longer conserved. Physically, this cannot happen for the electric charge, but it may happen for other charges such as the lepton number. Indeed, if we take the neutrino field to be a LH Weyl spinor field and give it a small Majorana mass, it would break the lepton number conservation and allow for the $\Delta L = \pm 2$ processes such as the neutrino-less double beta decay

$$\text{nucleus}(A,Z) \;\to\; \text{nucleus}(A,Z+2) \;+\; 2e^- \;+\; \text{no neutrinos.} \tag{56}$$

The rate for such a process is proportional to $m^2(\nu_e)$ and is expected to be very slow, which makes neutrino-less double beta decays very hard to detect. As of now, no such decays have been reliably observed, but several experimental groups are looking hard for them. If they succeed, this would give a direct measurement of the neutrino mass independent from the neutrino oscillations.

A non-zero neutrino mass, no matter how small, makes neutrinos spin $= \frac{1}{2}$ particles with two distinct spin states. At the same time, a Majorana mass term mixes breaks the phase symmetry associated with the lepton number, which causes neutrinos to mix up with the antineutrinos. In particular, a slow neutrino (moving with speed $v \ll c$) — if we could ever detect it experimentally — is an inherently neutral particle identical to an equally slow antineutrino. On the other hand, for the ultra-relativistic neutrinos with $E \gg m$, the mixture between the two chiralities of the Majorana spinor becomes negligible, with the $\left|\lambda = -\frac{1}{2}\right\rangle$ helicity state being a quantum of the ψ_L with very little admixture of the $\psi_R = \sigma_2 \psi_L^*$, while the $\left|\lambda = +\frac{1}{2}\right\rangle$ state is a quantum of $\psi_L^* = -\sigma_2 \psi_R$ with a very small admixture of the ψ_L. Consequently, for an ultra-relativistic neutrino/antineutrino, the species follows from the helicity: the $\left|\lambda = -\frac{1}{2}\right\rangle$ state is a neutrino while the $\left|\lambda = +\frac{1}{2}\right\rangle$ state is an antineutrino.

Parity and CP Symmetries

The **parity** or *space reflection* symmetry is a discrete Lorentz symmetry which inverts the space coordinates but not the time,

$$\mathbf{P}: \ (t, \mathbf{x}) \ \to \ (+t, -\mathbf{x}). \tag{57}$$

Since $\mathbf{p}^2 = 1$, its eigenstates are either P-even (unchanged under parity) or P-odd (flip sign). In terms of 3D scalars, vectors, *etc.*:

- A *true scalar* is P-even, for example time or energy, $\mathbf{P}: E \to +E$.
- A *pseudoscalar* is P-odd, for example helicity, $\mathbf{P}: \lambda \to -\lambda$.
- A *polar vector* (sometimes called true vector) is P-odd, for example velocity, momentum, force, or electric field, $\mathbf{P}: \vec{p} \to -\vec{p}$, $\mathbf{P}: \vec{E} \to -\vec{E}$, *etc.*
- An *axial vector* (sometimes called a pseudovector) is P-even, for example angular momentum or magnetic field, $\mathbf{P}: \vec{J} \to +\vec{J}$, $\mathbf{P}: \vec{B} \to +\vec{B}$, *etc.*

- An n-index tensor is called a *true tensor* if its parity is $P = (-1)^n$ and a *pseudotensor* if is parity is $P = -(-1)^n$. For example, the stress-tensor T^{ij} is P-even, $\mathbf{P} : T^{ij} \to +T^{ij}$, so it's a true tensor with $P = +1 = (-1)^2$.

In the Hilbert space of a quantum field theory, $\hat{\mathbf{P}}$ is a unitary operator which squares to 1,

$$\hat{\mathbf{P}}^2 = 1 \quad \Longrightarrow \quad \hat{\mathbf{P}}^{-1} = \hat{\mathbf{P}} = \hat{\mathbf{P}}^\dagger, \tag{58}$$

and acts on one-particle states according to

$$\hat{\mathbf{P}} \left| \mathbf{p}, s \right\rangle = \pm \left| -\mathbf{p}, +s \right\rangle . \tag{59}$$

The overall $\pm$ sign in this formula is the *inherent parity* of the particle species. For example, the π mesons are P-odd, thus for the pion states

$$\hat{\mathbf{P}} \left| \pi(\mathbf{p}) \right\rangle = - \left| \pi(-\mathbf{p}) \right\rangle . \tag{60}$$

Consequently, the pion field $\Phi_\pi(x)$ is pseudoscalar rather than true scalar: It transforms like a scalar under continuous Lorentz symmetries, but parity changes its sign:

$$\text{for } (t', \mathbf{x}') = (+t, -\mathbf{x}), \quad \Phi'(x') = -\Phi(x). \tag{61}$$

In your next homework#8 you should learn that the parity symmetry acts on the Dirac spinor fields according to

$$\text{for } (t', \mathbf{x}') = (+t, -\mathbf{x}), \quad \Psi'(x') = \pm\gamma^0\Psi(x), \tag{62}$$

and consequently *the particles and the antiparticles have opposite intrinsic parities,*

$$\hat{\mathbf{P}} \left| f(\mathbf{p}, s) \right\rangle = \pm \left| f(-\mathbf{p}, +s) \right\rangle \quad \text{while} \quad \hat{\mathbf{P}} \left| \bar{f}(\mathbf{p}, s) \right\rangle = \mp \left| \bar{f}(-\mathbf{p}, +s) \right\rangle \tag{63}$$

You should also learn that the intrinsic parity of a bound state of a fermion and antifermion — for example a $q\bar{q}$ meson — depends on the orbital angular momentum L but not on the net spin S; specifically

$$P \;=\; (-1)^{L+1}. \tag{64}$$

For example, both π^0 and ρ^0 mesons are quark-antiquark bound states with $L=0$, so both of these mesons are P-odd. Specifically, the pion is a pseudoscalar $J^P=0^-$ while the rho meson is a polar vector $J^P=1^-$.

The parity **P** and the charge conjugation **C** are exact symmetries of the strong and electromagnetic interactions. But back in 1956, the Wu experiment showed that the weak interaction do not have parity symmetry, and the follow-up experiments show that they also do not respect the charge conjugation symmetry. Instead, the weak interactions break both **P** and **C** symmetries in a maximal way: in the $E\gg m$ limit, the weak interactions couple only to the left-handed quarks and leptons but not to their right-handed counterparts; and for the antiparticles its the other way around: the weak interactions couple only to the right-handed antiquarks and antileptons but not to the left-handed antiquarks or antileptons.

The way weak interactions break **P** and **C** symmetries suggest that they do respect the combined **CP** symmetry: simultaneous space reflection and charge reversal. However, the 1964 Cronin and Fitch experiment showed that the combined **CP** symmetry is violated in decays of neutral K-mesons. Many years later, similar **CP** violations were discovered in the B-meson decays. I shall explain the **CP** violation in some detain in December. But for the moment, let me simply state that the combined **CP** is a good *approximate* symmetry of the weak interactions, although it is not an exact symmetry.

The best way to see how the weak interactions break separate **P** and **C** symmetries but are almost invariant under the combined **CP** is in terms of the Weyl fermion fields for the quarks and the leptons. In the Weyl convention, the parity action (62) on a Dirac spinor translates to

$$\Psi'_D(x') \;=\; \pm\gamma^0\Psi_D(x) \quad\Longrightarrow\quad \begin{pmatrix}\psi'_L(x')\\ \psi'_R(x')\end{pmatrix} \;=\; \pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_L(x')\\ \psi_R(x')\end{pmatrix} \tag{65}$$

and hence

$$\psi'_L(x') \;=\; \pm\psi_R(x), \quad \psi'_R(x') \;=\; \pm\psi_L(x). \tag{66}$$

Thus, besides reversing $\mathbf{x} \to -\mathbf{x}$, *the parity exchanges the LH and the RH Weyl spinors with each other.*

Now consider the charge conjugation symmetry and its action (8) on the Dirac spinor fields. The the Weyl spinor language, **C** acts as

$$\Psi'_D(x) \;=\; \pm\gamma^2\Psi^*_D(x) \quad\Longrightarrow\quad \begin{pmatrix}\psi'_L(x')\\ \psi'_R(x')\end{pmatrix} \;=\; \pm\begin{pmatrix}0 & \sigma_2\\ -\sigma_2 & 0\end{pmatrix}\begin{pmatrix}\psi^*_L(x')\\ \psi^*_R(x')\end{pmatrix} \tag{67}$$

and hence

$$\psi'_L(x) \;=\; \pm\sigma_2\psi^*_R(x), \quad \psi'_R(x) \;=\; \mp\sigma_2\psi^*_L(x). \tag{68}$$

Thus, apart from the $\pm\sigma_2$ factors, *the charge conjugations exchanges the LH and the RH Weyl spinors with each other's complex conjugates.*

We see that both parity and charge conjugation mix the two Weyl spinors with each other, so unless a field theory has both $\psi_L(x)$ and $\psi_R(x)$, we cannot even define the way **P** and **C** act on the fermions, let alone have the action $S = \int\mathcal{L}d^4x$ be invariant under these symmetries. However, the combined **CP** symmetry does not mix the $\psi_L(x)$ and the $\psi_R(x)$ with each other but only with their own conjugates. Indeed, combining eqs. (66) and (68), we see that the **CP** acts on the Weyl spinors as

$$\mathbf{CP}:(t',\mathbf{x}') \;=\; (+t,-\mathbf{x}), \quad \psi'_L(x') \;=\; \pm\sigma_2\psi^*_L(x), \quad \psi'_R(x') \;=\; \mp\sigma_2\psi^*_R(x). \tag{69}$$

Consequently, field theories may have a **CP** symmetry even if they have only the $\psi_L(x)$ field but no $\psi_R(x)$ field or the other way around. Note: I say 'may have' rather than 'have' because eq. (69) gives us a consistent way the symmetry acts on the fields but it does not guarantee that the action $S = \int\mathcal{L}d^4x$ would be invariant under this symmetry. We shall deal with this issue later in class.

For the moment, let's consider the weak interactions. Although all the quarks and all the leptons (except maybe neutrinos) have both LH and RH components $\psi_L(x)$ and $\psi_R(x)$, the $W^\pm_\mu(x)$ fields which mediate the weak interactions couple only to the LH Weyl spinors of the quarks and leptons but not to the RH Weyl spinors. Consequently, the weak interactions maximally break **P** and **C** symmetries which exchange the ψ_L fields with the ψ_R or the ψ_R^*. On the other hand, the combined **CP** symmetry which exchanges the ψ_L with its own conjugate and also W_μ^+ with W_μ^- might be a good symmetry of the weak interactions, depending on the details of the W–quark and W–lepton interactions. As it happens, the W–quark interactions are approximately — but not exactly — **CP** invariant, and that's why most weak decays are approximately **CP** symmetric, but some second-order processes like kaon oscillations manifest **CP** violation. I shall return to this subject in December.

Time reversal and CPT Theorem

Another discrete Lorentz symmetry is the time reversal, $(t,\mathbf{x}) \to (-t,+\mathbf{x})$. Physically, this symmetry is interpreted as *motion reversal* rather than literally making the time run back, hence an extra minus sign in the transformation rule for the energy momentum p^μ:

$$\mathbf{T} : (E,\mathbf{p}) \;\to\; (+E,-\mathbf{p}) \text{ rather than } (E,\mathbf{p}) \;\to\; (-E,+\mathbf{p}). \tag{70}$$

In quantum mechanics, having $t \to -t$ while $E \to +E$ turns the Schrödinger phases $\exp(-iEt/\hbar)$ into $\exp(+iEt/\hbar)$, which means that the symmetry operator $\hat{\mathbf{T}}$ in the Hilbert space should be *anti-unitary* rather than unitary:

$$\begin{aligned} \hat{\mathbf{T}}c \;&=\; c^*\hat{\mathbf{T}} \quad \text{for a complex number } c, \\ \text{and} \quad \big(\hat{\mathbf{T}}\,|\text{state}\#1\rangle\big)^\dagger\big(\hat{\mathbf{T}}\,|\text{state}\#1\rangle\big) \;&=\; \langle\text{state}\#1|\text{state}\#2\rangle^* \;\text{ instead of }\; \langle\text{state}\#1|\text{state}\#2\rangle\,. \end{aligned} \tag{71}$$

I wish I had time to discuss the time-reversal symmetry in any detail, but I do not. Instead, let me refer you to J. J. Sakurai's book *Modern Quantum Mechanics,* where §3.10 explores the time-reversal symmetry in quantum mechanics. Also, please read in the Peskin & Schroeder textbook the section about the time reversal symmetry and how it acts on the fermionic fields, namely §3.6 and especially pages 67–69.

The time reversal symmetry **T** may be combined with the parity **P**, the charge conjugation **C**, or with both of them. Of particular interest is the **CPT** symmetry which combines time reversal, space reversal, and charge reversal, all at the same time.

CPT theorem: All legitimate relativistic quantum field theories have exact **CPT** symmetry. More technically, any quantum field theory (or a classical field theory) which has: (1) continuous Lorentz symmetry, (2) Lorentz-invariant vacuum state, (3) Hermitian Hamiltonian operator, and (4) positive particle energies must have exact **CPT** symmetry.

Here are some down-to-experiment consequences of the **CPT** theorem:

- A particle and the corresponding antiparticle must have exactly the same mass.
- For unstable particles, the net decay rate of the particle P and of the antiparticle $\overline{P}$ are exactly equal,

$$\Gamma_{\text{total}}(P \to \text{anything}) \;=\; \Gamma_{\text{total}}(\overline{P} \to \text{anything}). \tag{72}$$

∗ On the other hand, in the absence of **C** or **CP** symmetries, the branching ratios for the P and $\overline{P}$ decays into specific final particles could be different,

$$\text{Br}(P \to \text{specific } X + Y + \cdots) \;\neq\; \text{Br}(\overline{P} \to \text{specific } \overline{X} + \overline{Y} + \cdots) \tag{73}$$

∗ Similar rules apply to the scattering cross-sections:

$$\sigma_{\text{total}}(A + B \to \text{anything}) \;=\; \sigma_{\text{total}}(\overline{A} + \overline{B} \to \text{anything}), \tag{74}$$

but in the absence of **C** or **CP** symmetries, the partial cross-sections for specific final-state particles could be different,

$$\sigma(A + B \to \text{specific } X + Y + \cdots) \;\neq\; \sigma(\overline{A} + \overline{B} \to \text{specific } \overline{X} + \overline{Y} + \cdots). \tag{75}$$

When decays or scattering cross-sections with asymmetric branching ratios happen in an out-of-thermal-equilibrium gas or plasma, that gas may end up with unequal numbers of particles and antiparticles even if initially their numbers were equal. In particular, the Early Universe probably had exactly equal numbers of quarks and antiquarks, hence zero net baryon number, but today we have more baryons than antibaryons, thus $B_{\text{net}} > 0$. Back in 1969, Andrey Sakharov came up with 3 criteria for a process which can build up this baryon excess:

- Baryon number non-conservation. Or rather, a B-changing process which was operated in the early Universe filled with a hod dense plasma but does not operate in the present-day environment.
- Broken **C** and **CP** symmetries to allow asymmetric branching ratios (73) and (75).
- The process violating the B, **C**, and **CP** symmetries must operate in an environment which is out of thermal equilibrium. (Otherwise, we would have detailed balance, and every B change due to some reaction would be canceled by the reverse reaction.)

Since then, physicists came up with many specific models of *baryogenesys*, all of them upholding the Sakharov's criteria. I wish I had time to review this fascinating subject in class, but the time is short, and this set of notes is already too long, so let me stop here.