

DIMENSIONAL REGULARIZATION

The dimensional regularization of ultraviolet divergences involves analytic continuation of the Euclidean momentum integrals to momentum spaces of non-integer dimensions $D < 4$ — which makes the integrals finite — and then taking the limit $D \rightarrow 4$ (from below). Thus,

$$\int_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4} f(k_E^\nu) = \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} f(k_E^\nu), \quad (1)$$

where μ is the reference energy scale at which the spherical momentum-space shell dk_D^{rad} has the same volume in D dimensions as in 4 dimensions. At much larger loop momenta, the dk_D^{rad} shell's volume becomes smaller in $D < 4$ dimensions than in 4 dimensions:

$$\begin{aligned} d^4 k_E \sim (k_E^{\text{rad}})^3 dk_E^{\text{rad}} &\longrightarrow \mu^{4-D} \times (k_E^{\text{rad}})^{D-1} dk_E^{\text{rad}} = \left(\frac{\mu}{k_E^{\text{rad}}} \right)^{4-D} \times (k_E^{\text{rad}})^3 dk_E^{\text{rad}} \\ &\ll (k_E^{\text{rad}})^3 dk_E^{\text{rad}}, \end{aligned} \quad (2)$$

and that's what regularized the UV divergence of the integral (1).

Let's take a closer look at the UV-regulating factor (marked in red in eq. (2)). For $D = 4 - 2\epsilon$,

$$\left(\frac{\mu}{k_E^{\text{rad}}} \right)^{4-D=2\epsilon} = \left(\frac{k_E^2}{\mu^2} \right)^{-\epsilon} = \exp\left(-\epsilon \times \log \frac{k_E^2}{\mu^2}\right), \quad (3)$$

which becomes small when

$$\log \frac{k_E^2}{\mu^2} \sim \frac{1}{\epsilon} \implies k_E^2 \sim \mu^2 \times \exp(1/\epsilon). \quad (4)$$

Thus, the effective UV cutoff scale² in dimensional regularization is

$$\Lambda_{\text{DR}}^2 = \mu^2 \times \exp(1/\epsilon) \gg \mu^2. \quad (5)$$

In practice, one usually set the reference energy scale μ in the ball park of the energy scale of the amplitude in question, for example $\mu \sim |q_{\text{net}}|$; consequently, for $\epsilon \rightarrow +0$ we have $\Lambda_{\text{DR}} \gg \mu$ and hence $\Lambda_{\text{DR}} \gg$ energy scale of the amplitude.

Now consider a generic logarithmically divergent momentum integral; for most regularization schemes, this means

$$\text{regulated integral} = (\text{constant } C) \times \log \frac{\Lambda^2}{m^2} + \text{finite}. \quad (6)$$

For the dimensional regularization, the effective UV cutoff scale is as in eq. (5), so we expect

$$\text{regulated integral} = (\text{same constant } C) \times \left(\frac{1}{\epsilon} + \log \frac{\mu^2}{m^2} \right) + \text{finite}. \quad (7)$$

Thus, we may identify the coefficient C of the $(1/\epsilon)$ pole obtaining from dimensional regularization with the coefficient of $\log \Lambda^2$ in the other regularization schemes.

Integrals over Momentum Spaces of Non-Integer Dimensions

Before we can use dimensional regularization, we need to learn how to perform integrals over (Euclidean) momentum spaces of non-integer dimensions D . Let's start with the Gaussian integrals

$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-tk_E^2). \quad (8)$$

For any integer dimension D , $k_E^2 = k_1^2 + k_2^2 + \dots + k_D^2$, hence

$$\exp(-tk_E^2) = \prod_{i=1}^D \exp(-tk_i^2) \quad (9)$$

and therefore

$$\begin{aligned} \int \frac{d^D k_E}{(2\pi)^D} \exp(-tk_E^2) &= \prod_{i=1}^D \int_{-\infty}^{+\infty} \frac{dk_i}{2\pi} \exp(-tk_i^2) \\ &= \left[\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \exp(-tk^2) \right]^D \\ &= \left[\frac{1}{2\pi} \times \sqrt{\frac{\pi}{t}} = \frac{1}{\sqrt{4\pi t}} \right]^D \\ &= (4\pi t)^{-D/2}. \end{aligned} \quad (10)$$

Let's analytically continue this formula to the non-integer D . In other words, we let

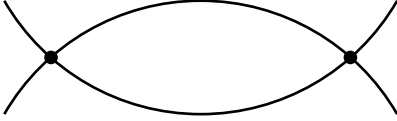
$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-tk_E^2) = (4\pi t)^{-D/2} \quad (11)$$

for any D , integer or non-integer, real or complex. For non-integer D this formula maybe thought as a *definition* of the Gaussian integral over a non-integer-dimensional space.

As to the non-Gaussian momentum integrals, we should re-express them in terms of Gaussian integrals and then use eq. (11) for non-integer D . For example, consider the dimensionally regulated momentum integral

$$I = \int_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2} = \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + \Delta]^2} \quad (12)$$

which appears in the context of the one-loop Feynman diagram



$$\mathcal{F}(t) = \frac{\lambda^2}{2} \int_0^1 dx \int_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta(x)]^2} \quad (13)$$

for $\Delta(x) = m^2 - tx(1-x)$. Using the Γ -function integral

$$\int_0^\infty dt t^{n-1} \times \exp(-t(k_E^2 + \Delta)) = \frac{\Gamma(n)}{[k_E^2 + \Delta]^n} \quad (14)$$

for $n = 2$, we let

$$\frac{1}{[k_E^2 + \Delta]^2} = \frac{1}{\Gamma(2) = 1! = 1} \times \int_0^\infty dt t \times \exp(-t(k_E^2 + \Delta)) \quad (15)$$

and consequently

$$\begin{aligned}
\int \frac{d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + \Delta]^2} &= \int \frac{d^D k_E}{(2\pi)^D} \int_0^\infty dt t \times \exp(-t(k_E^2 + \Delta)) \\
&\langle\langle \text{changing the order of integration} \rangle\rangle \\
&= \int_0^\infty dt t e^{-t\Delta} \times \int \frac{d^D k_E}{(2\pi)^D} e^{-tk_E^2} \\
&\langle\langle \text{using eq. (11)} \rangle\rangle \\
&= \int_0^\infty dt t e^{-t\Delta} \times (4\pi t)^{-D/2} = (4\pi)^{-D/2} \int_0^\infty dt t^{1-(D/2)} \times e^{-t\Delta} \\
&= (4\pi)^{-D/2} \times \Gamma(2 - (D/2)) \Delta^{(D/2)-2}.
\end{aligned} \tag{16}$$

Note that on the penultimate line here, the integrand behaves as $t^{1-(D/2)}$ for $t \rightarrow 0$. Consequently, the integral converges whenever (this power of t) > -1 , which means $D < 4$. Or for complex D , whenever $\text{Re}(D) < 4$. Physically, the $t \rightarrow 0$ limit corresponds to $k_E^2 \rightarrow \infty$, so the convergence/divergence of the $\int dt$ integral at $t \rightarrow 0$ corresponds to the UV convergence/divergence of the original momentum integral.

Anyhow, for $D = 4 - 2\epsilon$ eq. (16) becomes

$$\mu^{4-D} \times \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + \Delta]^2} = \frac{(4\pi\mu^2)^\epsilon}{16\pi^2} \times \Gamma(\epsilon) \times \Delta^{-\epsilon} \tag{17}$$

and hence

$$\mathcal{F}(t) = \frac{\lambda^2}{32\pi^2} \int_0^1 dx \Gamma(\epsilon) \left(\frac{4\pi\mu^2}{\Delta(x)} \right)^\epsilon. \tag{18}$$

Note that this is a finite formula for $\epsilon > 0$ (*i.e.*, for $D < 4$), but it becomes singular in the $\epsilon \rightarrow 0$ limit because the $\Gamma(\epsilon)$ function has a pole at $\epsilon = 0$.

Let's take a closer look at this pole using $\Gamma(x+1) = x \times \Gamma(x)$. In particular, for $x = \epsilon \rightarrow 0$,

$$\begin{aligned}
\Gamma(\epsilon) &= \frac{\Gamma(\epsilon + 1)}{\epsilon} = \frac{1}{\epsilon} \left(\Gamma(1) + \epsilon \times \Gamma'(1) + \frac{\epsilon^2}{2} \Gamma''(1) + \dots \right) \\
&= \frac{1}{\epsilon} - \gamma_E + \frac{\pi^2 + 6\gamma_E^2}{12} \times \epsilon + O(\epsilon^2)
\end{aligned} \tag{19}$$

where $\gamma_E \approx 0.5772$ is the [Euler–Mascheroni constant](#). At the same time,

$$\left(\frac{4\pi\mu^2}{\Delta(x)}\right)^\epsilon = \exp\left(\epsilon \times \log \frac{4\pi\mu^2}{\Delta(x)}\right) = 1 + \epsilon \times \log \frac{4\pi\mu^2}{\Delta(x)} + \frac{\epsilon^2}{2} \times \log^2 \frac{4\pi\mu^2}{\Delta(x)} + O(\epsilon)^3, \quad (20)$$

hence

$$\Gamma(\epsilon) \times \left(\frac{4\pi\mu^2}{\Delta(x)}\right)^\epsilon = \frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{\Delta(x)} + O(\epsilon). \quad (21)$$

In dimensional regularization, positive powers of $\epsilon \rightarrow 0$ correspond to negative powers of $\log \Lambda_{\text{UV}}^2 \rightarrow \infty$. And although such negative powers of $\log \Lambda_{\text{UV}}^2$ go to zero much slower than the negative powers of the Λ_{UV}^2 itself, they do eventually go to zero in the very-large-UV-cutoff-scale limit. Consequently, **in dimensional regularization we neglect all *positive* powers of ϵ in various amplitudes** (but only in the net product of all the factors). Thus, in eq. (18) we approximate

$$\Gamma(\epsilon) \times \left(\frac{4\pi\mu^2}{\Delta(x)}\right)^\epsilon \xrightarrow{\epsilon \rightarrow 0} \frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{\Delta(x)} \quad (22)$$

and hence

$$\mathcal{F}_{\text{DR}}(t) = \frac{\lambda^2}{32\pi^2} \int_0^1 dx \left(\frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{\Delta(x)} \right). \quad (23)$$

Finally, using

$$\log \frac{4\pi\mu^2}{\Delta(x)} = \log \frac{4\pi\mu^2}{m^2} - \log \frac{\Delta(x) = m^2 - tx(1-x)}{m^2} \quad (24)$$

we arrive at

$$\begin{aligned} \mathcal{F}_{\text{DR}}(t) &= \frac{\lambda^2}{32\pi^2} \left(\frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{m^2} - \int_0^1 dx \log \frac{m^2 - tx(1-x)}{m^2} \right) \\ &= \frac{\lambda^2}{32\pi^2} \left(\frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{m^2} - J(t/m^2) \right). \end{aligned} \quad (25)$$

In class we have evaluated the same one-loop diagram (13) using Wilson's hard-edge

cutoff and got

$$\mathcal{F}(t) = \frac{\lambda^2}{32\pi^2} \left(\log \frac{\Lambda_{\text{HE}}^2}{m^2} - 1 - J(t/m^2) \right). \quad (26)$$

Likewise, in your [homework#13](#) you should have obtained

$$\begin{aligned} \mathcal{F}(t) &= \frac{\lambda^2}{32\pi^2} \left(\log \frac{\Lambda_{\text{PV}}^2}{m^2} - J(t/m^2) \right) \\ &= \frac{\lambda^2}{32\pi^2} \left(\log \frac{\Lambda_{\text{HD}}^2}{m^2} - 2 - J(t/m^2) \right) \end{aligned} \quad (27)$$

for the respectively Pauli–Villars and higher-derivative UV regulators. Consequently, all these cutoffs yield exactly the same result provided we identify

$$\log \Lambda_{\text{HE}}^2 - 1 = \log \Lambda_{\text{PV}}^2 = \log \Lambda_{\text{HD}}^2 - 2, \quad (28)$$

or equivalently

$$\Lambda_{\text{HE}}^2 = \exp(1) \times \Lambda_{\text{PV}}^2, \quad \Lambda_{\text{HD}}^2 = \exp(2) \times \Lambda_{\text{PV}}^2. \quad (29)$$

Likewise, the dimensional regularization's result (25) becomes similar to that of all the other cutoffs when we identify

$$\frac{1}{\epsilon} - \gamma_E + \log(4\pi\mu^2) = \log \Lambda_{\text{HE}}^2 - 1 = \log \Lambda_{\text{PV}}^2 = \log \Lambda_{\text{HD}}^2 - 2, \quad (30)$$

or equivalently

$$\mu^2 \times \exp(1/\epsilon) = \frac{\exp(\gamma_E)}{4\pi} \times \Lambda_{\text{PV}}^2 = \frac{\exp(\gamma_E - 1)}{4\pi} \times \Lambda_{\text{HE}}^2 = \frac{\exp(\gamma_E - 2)}{4\pi} \times \Lambda_{\text{HD}}^2. \quad (31)$$