

# Dirac Matrices and Lorentz Spinors

**Background:** In 3D, the spinor  $j = \frac{1}{2}$  representation of the Spin(3) rotation group is constructed from the Pauli matrices  $\sigma^x$ ,  $\sigma^y$ , and  $\sigma^z$ , which obey both commutation and anticommutation relations

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \quad \text{and} \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij} \times \mathbf{1}_{2 \times 2}. \quad (1)$$

Consequently, the spin matrices

$$\mathbf{S} = -\frac{i}{4}\boldsymbol{\sigma} \times \boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\sigma} \quad (2)$$

commute with each other like angular momenta,  $[S^i, S^j] = i\epsilon^{ijk}S^k$ , so they represent the generators of the rotation group. In this spinor representation, the finite rotations  $R(\phi, \mathbf{n})$  are represented by

$$M(R) = \exp(-i\phi\mathbf{n} \cdot \mathbf{S}), \quad (3)$$

while the spin matrices themselves transform into each other as components of a 3-vector,

$$M^{-1}(R)S^iM(R) = R^{ij}S^j. \quad (4)$$

In this note, I shall generalize this construction to the *Dirac spinor* representation of the Lorentz symmetry Spin(3, 1).

**The Dirac Matrices**  $\gamma^\mu$  generalize the anti-commutation properties of the Pauli matrices  $\sigma^i$  to the 3 + 1 Minkowski dimensions:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbf{1}_{4 \times 4}. \quad (5)$$

The  $\gamma^\mu$  are  $4 \times 4$  matrices, but there are several different conventions for their specific form. In my class I shall follow the same convention as the Peskin & Schroeder textbook, namely

the Weyl convention where in  $2 \times 2$  block notations

$$\gamma^0 = \begin{pmatrix} 0_{2 \times 2} & 1_{2 \times 2} \\ 1_{2 \times 2} & 0_{2 \times 2} \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} \mathbf{0}_{2 \times 2} & +\boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0}_{2 \times 2} \end{pmatrix}. \quad (6)$$

Note that  $\gamma^0$  is an hermitian matrix while  $\gamma^1, \gamma^2$ , and  $\gamma^3$  are anti-hermitian matrices. Apart from that, the specific forms of the matrices are not important, the Physics follows from the anti-commutation relations (5).

**The Lorentz spin matrices** generalize  $\mathbf{S} = -\frac{i}{2}\boldsymbol{\sigma} \times \boldsymbol{\sigma}$  rather than  $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ . In 4D, the vector product becomes the antisymmetric tensor product, so we define

$$S^{\mu\nu} \stackrel{\text{def}}{=} -S^{\nu\mu} \stackrel{\text{def}}{=} \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \quad (7)$$

Consequently,

$$\gamma^\mu \gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = g^{\mu\nu} \times 1_{4 \times 4} - 2iS^{\mu\nu}. \quad (8)$$

Thanks to the anti-commutation relations (5) for the  $\gamma^\mu$  matrices, the  $S^{\mu\nu}$  obey the commutation relations of the Lorentz generators  $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$ . Moreover, the commutation relations of the spin matrices  $S^{\mu\nu}$  with the Dirac matrices  $\gamma^\mu$  are similar to the commutation relations of the  $\hat{J}^{\mu\nu}$  with a Lorentz vector such as  $\hat{P}^\mu$ .

**Lemma:**

$$[\gamma^\lambda, S^{\mu\nu}] = ig^{\lambda\mu}\gamma^\nu - ig^{\lambda\nu}\gamma^\mu. \quad (9)$$

Proof: In light of eq. (8) and the fact that the unit matrix commutes with everything, we have

$$[X, S^{\mu\nu}] = \frac{i}{2}[X, \gamma^\mu \gamma^\nu] \quad \text{for any matrix } X. \quad (10)$$

Moreover, the commutator on the RHS may often be obtained from the **Leibniz rules for the commutators or anticommutators:**

$$\begin{aligned} [A, BC] &= [A, B]C + B[A, C] = \{A, B\}C - B\{A, C\}, \\ \{A, BC\} &= [A, B]C + B\{A, C\} = \{A, B\}C - B[A, C]. \end{aligned} \quad (11)$$

In particular,

$$[\gamma^\lambda, \gamma^\mu \gamma^\nu] = \{\gamma^\lambda, \gamma^\mu\} \gamma^\nu - \gamma^\mu \{\gamma^\lambda, \gamma^\nu\} = 2g^{\lambda\mu} \gamma^\nu - 2g^{\lambda\nu} \gamma^\mu \quad (12)$$

and hence

$$[\gamma^\lambda, S^{\mu\nu}] = \frac{i}{2} [\gamma^\lambda, \gamma^\mu \gamma^\nu] = ig^{\lambda\mu} \gamma^\nu - ig^{\lambda\nu} \gamma^\mu. \quad (13)$$

*Quod erat demonstrandum.*

**Theorem:** The  $S^{\mu\nu}$  matrices commute with each other like Lorentz generators,

$$[S^{\kappa\lambda}, S^{\mu\nu}] = ig^{\lambda\mu} S^{\kappa\nu} - ig^{\lambda\nu} S^{\kappa\mu} - ig^{\kappa\mu} S^{\lambda\nu} + ig^{\kappa\nu} S^{\lambda\mu}. \quad (14)$$

Proof: Again, we use the Leibniz rule and eq. (8):

$$\begin{aligned} [\gamma^\kappa \gamma^\lambda, S^{\mu\nu}] &= \gamma^\kappa [\gamma^\lambda, S^{\mu\nu}] + [\gamma^\kappa, S^{\mu\nu}] \gamma^\lambda \\ &= \gamma^\kappa (ig^{\lambda\mu} \gamma^\nu - ig^{\lambda\nu} \gamma^\mu) + (ig^{\kappa\mu} \gamma^\nu - ig^{\kappa\nu} \gamma^\mu) \gamma^\lambda \\ &= ig^{\lambda\mu} (\gamma^\kappa \gamma^\nu = g^{\kappa\nu} - 2iS^{\kappa\nu}) - ig^{\lambda\nu} (\gamma^\kappa \gamma^\mu = g^{\kappa\mu} - 2iS^{\kappa\mu}) \\ &\quad + ig^{\kappa\mu} (\gamma^\nu \gamma^\lambda = g^{\lambda\nu} + 2iS^{\lambda\nu}) - ig^{\kappa\nu} (\gamma^\mu \gamma^\lambda = g^{\lambda\mu} + 2iS^{\lambda\mu}) \\ &= \cancel{ig^{\lambda\mu} g^{\kappa\nu}} - \cancel{ig^{\lambda\nu} g^{\kappa\mu}} + \cancel{ig^{\kappa\mu} g^{\lambda\nu}} - \cancel{ig^{\kappa\nu} g^{\lambda\mu}} \quad \langle\langle \text{note cancellation} \rangle\rangle \\ &\quad + 2g^{\lambda\mu} S^{\kappa\nu} - 2g^{\lambda\nu} S^{\kappa\mu} - 2g^{\kappa\mu} S^{\lambda\nu} + 2g^{\kappa\nu} S^{\lambda\mu} \\ &= 2g^{\lambda\mu} S^{\kappa\nu} - 2g^{\lambda\nu} S^{\kappa\mu} - 2g^{\kappa\mu} S^{\lambda\nu} + 2g^{\kappa\nu} S^{\lambda\mu} \end{aligned} \quad (15)$$

and therefore

$$[S^{\kappa\lambda}, S^{\mu\nu}] = \frac{i}{2} [\gamma^\kappa \gamma^\lambda, S^{\mu\nu}] = ig^{\lambda\mu} S^{\kappa\nu} - ig^{\lambda\nu} S^{\kappa\mu} - ig^{\kappa\mu} S^{\lambda\nu} + ig^{\kappa\nu} S^{\lambda\mu}. \quad (16)$$

*Quod erat demonstrandum.*

In light of this theorem, the  $S^{\mu\nu}$  matrices *represent* the Lorentz generators  $\hat{J}^{\mu\nu}$  in the 4-component spinor multiplet.

### Finite Lorentz transforms:

Any continuous Lorentz transform — a rotation, or a boost, or a product of a boost and a rotation — obtains from exponentiating an infinitesimal symmetry

$$X'^{\mu} = X^{\mu} + \epsilon^{\mu}_{\nu} X^{\nu} \quad (17)$$

where the infinitesimal  $\epsilon^{\mu}_{\nu}$  matrix becomes antisymmetric when both indices are raised,  $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ . For a finite continuous Lorentz transform, we start with a finite matrix  $\Theta^{\mu}_{\nu}$  which obeys  $(g\Theta)^{\top} = -(g\Theta)$  so that when we raise the second index it becomes antisymmetric,  $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$ . Then we exponentiate the transform (17) for  $\Theta^{\mu}_{\nu}$  in place of  $\epsilon^{\mu}_{\nu}$ , thus

$$X'^{\mu} = X^{\mu} + \Theta^{\mu}_{\nu} X^{\nu} + \frac{1}{2} \Theta^{\mu}_{\lambda} \Theta^{\lambda}_{\nu} X^{\nu} + \frac{1}{6} \Theta^{\mu}_{\lambda} \Theta^{\lambda}_{\rho} \Theta^{\rho}_{\nu} X^{\nu} + \dots \quad (18)$$

In other words,

$$\begin{aligned} X'^{\mu} &= L^{\mu}_{\nu} X^{\nu} \\ \text{for } L^{\mu}_{\nu} &= \delta^{\mu}_{\nu} + \Theta^{\mu}_{\nu} + \frac{1}{2} \Theta^{\mu}_{\lambda} \Theta^{\lambda}_{\nu} + \frac{1}{6} \Theta^{\mu}_{\lambda} \Theta^{\lambda}_{\kappa} \Theta^{\kappa}_{\nu} + \dots \\ &= (\exp(\Theta))^{\mu}_{\nu}. \end{aligned} \quad (19)$$

The Dirac spinor representation of the finite Lorentz transform (19) is the  $4 \times 4$  matrix

$$M_D(L) = \exp\left(-\frac{i}{2} \Theta_{\kappa\lambda} S^{\kappa\lambda}\right). \quad (20)$$

The group law for such matrices

$$\forall L_1, L_2 \in \text{SO}^+(3,1), \quad M_D(L_2 L_1) = M_D(L_2) M_D(L_1) \quad (21)$$

follows automatically from the  $S^{\mu\nu}$  satisfying the commutation relations (14) of the Lorentz generators, so I am not going to prove it. Instead, let me show that when the Dirac matrices  $\gamma^{\mu}$  are sandwiched between the  $M_D(L)$  and its inverse, they transform into each other as components of a Lorentz 4-vector,

$$M_D^{-1}(L) \gamma^{\mu} M_D(L) = L^{\mu}_{\nu} \gamma^{\nu}. \quad (22)$$

*This formula makes the Dirac equation transform covariantly under the Lorentz transforms.*

Proof: In light of the exponential form (20) of the matrix  $M_D(L)$  representing a finite Lorentz transform in the Dirac spinor multiplet, let's use the multiple commutator formula (AKA the [Hadamard Lemma](#)): for any 2 matrices  $F$  and  $H$ ,

$$\exp(-F)H \exp(+F) = H + [H, F] + \frac{1}{2} [[H, F], F] + \frac{1}{6} [[[H, F], F], F] + \dots \quad (23)$$

In particular, let  $H = \gamma^\mu$  while  $F = -\frac{i}{2} \Theta_{\kappa\lambda} S^{\kappa\lambda}$  so that  $M_D(L) = \exp(+F)$  and  $M_D^{-1}(L) = \exp(-F)$ . Consequently,

$$M_D^{-1}(L)\gamma^\mu M_D(L) = \gamma^\mu + [\gamma^\mu, F] + \frac{1}{2} [[\gamma^\mu, F], F] + \frac{1}{6} [[[\gamma^\mu, F], F], F] + \dots \quad (24)$$

where all the multiple commutators turn out to be linear combinations of the Dirac matrices. Indeed, the single commutator here is

$$[\gamma^\mu, F] = -\frac{i}{2} \Theta_{\kappa\lambda} [\gamma^\mu, S^{\kappa\lambda}] = -\frac{i}{2} \Theta_{\kappa\lambda} (i g^{\mu\kappa} \gamma^\lambda - i g^{\mu\lambda} \gamma^\kappa) = \Theta_{\kappa\lambda} g^{\mu\kappa} \gamma^\lambda = \Theta^\mu_\lambda \gamma^\lambda, \quad (25)$$

while the multiple commutators follow by iterating this formula:

$$[[\gamma^\mu, F], F] = \Theta^\mu_\lambda [\gamma^\lambda, F] = \Theta^\mu_\lambda \Theta^\lambda_\nu \gamma^\nu, \quad [[[\gamma^\mu, F], F], F] = \Theta^\mu_\lambda \Theta^\lambda_\rho \Theta^\rho_\nu \gamma^\nu, \dots \quad (26)$$

Combining all these commutators as in eq. (24), we obtain

$$\begin{aligned} M_D^{-1}\gamma^\mu M_D &= \gamma^\mu + [\gamma^\mu, F] + \frac{1}{2} [[\gamma^\mu, F], F] + \frac{1}{6} [[[\gamma^\mu, F], F], F] + \dots \\ &= \gamma^\mu + \Theta^\mu_\nu \gamma^\nu + \frac{1}{2} \Theta^\mu_\lambda \Theta^\lambda_\nu \gamma^\nu + \frac{1}{6} \Theta^\mu_\lambda \Theta^\lambda_\rho \Theta^\rho_\nu \gamma^\nu + \dots \\ &= \left( \delta^\mu_\nu + \Theta^\mu_\nu + \frac{1}{2} \Theta^\mu_\lambda \Theta^\lambda_\nu + \frac{1}{6} \Theta^\mu_\lambda \Theta^\lambda_\rho \Theta^\rho_\nu + \dots \right) \gamma^\nu \\ &\equiv L^\mu_\nu \gamma^\nu. \end{aligned} \quad (27)$$

*Quod erat demonstrandum.*

# Dirac Equation and Dirac Spinor Fields

## History:

Originally, the Klein–Gordon equation was thought to be the relativistic version of the Schrödinger equation — that is, an equation for the wave function  $\psi(\mathbf{x}, t)$  for one relativistic particle. But pretty soon this interpretation run into trouble with bad probabilities (negative or  $> 1$ ) when a particle travels through high potential barriers or deep potential wells. There were also troubles with relativistic causality, and a few other things.

Paul Adrien Maurice Dirac had thought that the source of all those troubles was the ugly form of relativistic Hamiltonian  $\hat{H} = \sqrt{\hat{\mathbf{p}}^2 + m^2}$  in the coordinate basis, and that he could solve all the problems with the Klein-Gordon equation by rewriting the Hamiltonian as a first-order differential operator

$$\hat{H} = \hat{\mathbf{p}} \cdot \vec{\alpha} + m\beta \implies \text{Dirac equation } i \frac{\partial \psi}{\partial t} = -i\vec{\alpha} \cdot \nabla \psi + m\beta \psi \quad (28)$$

where  $\alpha_1, \alpha_2, \alpha_3, \beta$  are matrices acting on a multi-component wave function. Specifically, all four of these matrices are Hermitian, square to 1, and anticommute with each other,

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\alpha_i, \beta\} = 0, \quad \beta^2 = 1. \quad (29)$$

Consequently

$$(\vec{\alpha} \cdot \hat{\mathbf{p}})^2 = \alpha_i \alpha_j \times \hat{p}_i \hat{p}_j = \frac{1}{2} \{\alpha_i, \alpha_j\} \times \hat{p}_i \hat{p}_j = \delta_{ij} \times \hat{p}_i \hat{p}_j = \hat{\mathbf{p}}^2, \quad (30)$$

and therefore

$$\hat{H}_{\text{Dirac}}^2 = (\vec{\alpha} \cdot \hat{\mathbf{p}} + \beta m)^2 = (\vec{\alpha} \cdot \hat{\mathbf{p}})^2 + \{\alpha_i, \beta\} \times \hat{p}_i m + \beta^2 \times m^2 = \hat{\mathbf{p}}^2 + 0 + m^2. \quad (31)$$

This, the Dirac Hamiltonian squares to  $\hat{\mathbf{p}}^2 + m^2$ , as it should for a relativistic particle.

The Dirac equation (28) turned out to be a much better description of a relativistic electron (which has spin =  $\frac{1}{2}$ ) than the Klein–Gordon equation. However, it did not resolve the troubles with relativistic causality or bad probabilities for electrons going through big potential differences  $e\Delta\Phi > 2m_e c^2$ . Those problems are not solvable in the context of a relativistic single-particle quantum mechanics but only in the quantum field theory.

### Modern point of view:

Today, we interpret the Dirac equation as the equation of motion for a Dirac spinor field  $\Psi(x)$ , — which comprises 4 complex component fields  $\Psi_\alpha(x)$  arranged in a column vector

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix}, \quad (32)$$

and transforms under the continuous Lorentz symmetries  $x'^\mu = L^\mu_\nu x^\nu$  according to

$$\Psi'(x') = M_D(L)\Psi(x). \quad (33)$$

The classical Euler–Lagrange equation of motion for the spinor field is the Dirac equation

$$i \frac{\partial}{\partial t} \Psi + i \vec{\alpha} \cdot \nabla \Psi - m \beta \Psi = 0. \quad (34)$$

To recast this equation in a Lorentz-covariant form, let

$$\beta = \gamma^0, \quad \alpha^i = \gamma^0 \gamma^i; \quad (35)$$

it is easy to see that if the  $\gamma^\mu$  matrices obey the anticommutation relations (5) then the  $\vec{\alpha}$  and  $\beta$  matrices obey the relations (29) and vice versa. Now let's multiply the whole LHS of the Dirac equation (34) by the  $\beta = \gamma^0$ :

$$0 = \gamma^0 \left( i \partial_0 + i \gamma^0 \vec{\gamma} \cdot \nabla - m \gamma^0 \right) \Psi(x) = \left( i \gamma^0 \partial_0 + i \gamma^i \partial_i - m \right) \Psi(x), \quad (36)$$

and hence

$$(i \gamma^\mu \partial_\mu - m) \Psi(x) = 0. \quad (37)$$

As expected from  $\hat{H}_{\text{Dirac}}^2 = \hat{\mathbf{p}}^2 + m^2$ , the Dirac equation for the spinor field implies the Klein–Gordon equation for each component  $\Psi_\alpha(x)$ . Indeed, if  $\Psi(x)$  obeys the Dirac equation,

then obviously

$$(-i\gamma^\nu\partial_\nu - m) \times (i\gamma^\mu\partial_\mu - m)\Psi(x) = 0, \quad (38)$$

but the differential operator on the LHS is equal to the Klein–Gordon  $m^2 + \partial^2$  times a unit matrix:

$$(-i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu - m) = m^2 + \gamma^\nu\gamma^\mu\partial_\nu\partial_\mu = m^2 + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\nu\partial_\mu = m^2 + g^{\mu\nu}\partial_\nu\partial_\mu. \quad (39)$$

**The Dirac equation (37) transforms covariantly under the Lorentz symmetries.** That is, whether  $\Psi(x)$  obeys the Dirac equation or not, the LHS  $(i\gamma^\mu\partial_\mu - m)\Psi(x)$  transform exactly like the spinor field  $\Psi(x)$  itself.

Proof: Note that since the Lorentz symmetries involve the  $x^\mu$  coordinates as well as the spinor field components, the LHS of the Dirac equation becomes

$$(i\gamma^\mu\partial'_\mu - m)\Psi'(x') \quad (40)$$

where

$$\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \times \frac{\partial}{\partial x^\nu} = (L^{-1})^\nu{}_\mu \times \partial_\nu. \quad (41)$$

Consequently,

$$\partial'_\mu\Psi'(x') = (L^{-1})^\nu{}_\mu \times M_D(L) \partial_\nu\Psi(x) \quad (42)$$

and hence

$$\gamma^\mu\partial'_\mu\Psi'(x') = (L^{-1})^\nu{}_\mu \times \gamma^\mu M_D(L) \partial_\nu\Psi(x). \quad (43)$$

But according to eq. (24),

$$\begin{aligned} M_D^{-1}(L)\gamma^\mu M_D(L) = L^\mu{}_\nu\gamma^\nu &\implies \gamma^\mu M_D(L) = L^\mu{}_\nu \times M_D(L)\gamma^\nu \\ &\implies (L^{-1})^\nu{}_\mu \times \gamma^\mu M_D(L) = M_D(L)\gamma^\nu, \end{aligned} \quad (44)$$

so

$$\gamma^\mu\partial'_\mu\Psi'(x') = M_D(L) \times \gamma^\nu\partial_\nu\Psi(x). \quad (45)$$



Altogether,

$$\begin{aligned} \Psi(x) &\xrightarrow{\text{Lorentz}} \Psi'(x') = M_D(L)\Psi(x), \\ (i\gamma^\mu\partial_\mu - m)\Psi(x) &\xrightarrow{\text{Lorentz}} (i\gamma^\mu\partial'_\mu - m)\Psi'(x') = M_D(L) \times (i\gamma^\mu\partial_\mu - m)\Psi(x), \end{aligned} \quad (46)$$

which proves covariance of the Dirac equation. *Quod erat demonstrandum.*

## Dirac Lagrangian

The Dirac equation is a first-order differential equation, so to obtain it as an Euler–Lagrange equation, we need a Lagrangian which is linear rather than quadratic in the spinor field’s derivatives. Thus, we want

$$\mathcal{L} = \bar{\Psi} \times (i\gamma^\mu\partial_\mu - m)\Psi \quad (47)$$

where  $\bar{\Psi}(x)$  is some kind of a conjugate field to the  $\Psi(x)$ . Since  $\Psi$  is a complex field, we treat  $\Psi$  and  $\bar{\Psi}$  as linearly-independent from each other, so the Euler–Lagrange equation for the  $\bar{\Psi}$  immediately gives us the Dirac equation for the  $\Psi(x)$  field,

$$0 = \frac{\partial\mathcal{L}}{\partial\bar{\Psi}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi})} = (i\gamma^\nu\partial_\nu - m)\Psi - \partial_\mu(0). \quad (48)$$

The conjugate field also obeys a Dirac-like equation

$$0 = \frac{\partial\mathcal{L}}{\partial\Psi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} = -m\bar{\Psi} - i\partial_\mu\bar{\Psi}\gamma^\mu. \quad (49)$$

But before we worry about the field equation for the  $\bar{\Psi}(x)$  or even identify its relation to the  $\Psi(x)$  field, let’s consider the Lorentz transformation rule for the  $\bar{\Psi}(x)$ . To keep the action  $S = \int d^4x\mathcal{L}$  Lorentz-invariant, the Lagrangian (47) should transform as a Lorentz scalar,  $\mathcal{L}'(x') = \mathcal{L}(x)$ . In light of eq. (20) for the  $\Psi(x)$  field and covariance (46) of the Dirac equation, the conjugate field  $\bar{\Psi}(x)$  should transform according to

$$\bar{\Psi}'(x') = \bar{\Psi}(x) \times M_D^{-1}(L) \implies \mathcal{L}'(x') = \mathcal{L}(x). \quad (50)$$

Note that the  $M_D(L)$  matrix is generally not unitary, so the inverse matrix  $M_D^{-1}(L)$  in eq. (50) is different from the hermitian conjugate  $M_D^\dagger(L)$ . Consequently, the conjugate

field  $\bar{\Psi}(x)$  cannot be identified with the hermitian conjugate field  $\Psi^\dagger(x)$ , since the latter transforms to

$$\Psi'^\dagger(x') = \Psi^\dagger(x) \times M_D^\dagger(L) \neq \Psi^\dagger(x) \times M_D^{-1}(L). \quad (51)$$

Instead of the hermitian conjugate, we are going to use the Dirac conjugate spinor, see below.

### Dirac conjugates:

Let  $\Psi$  be a 4-component Dirac spinor and  $\Gamma$  be any  $4 \times 4$  matrix; we *define* their Dirac conjugates according to

$$\bar{\Psi} = \Psi^\dagger \times \gamma^0, \quad \bar{\Gamma} = \gamma^0 \times \Gamma^\dagger \times \gamma^0. \quad (52)$$

Thanks to  $\gamma^0 \gamma^0 = 1$ , the Dirac conjugates behave similarly to hermitian conjugates or transposed matrices:

- For a product of 2 matrices,  $\overline{(\Gamma_1 \times \Gamma_2)} = \bar{\Gamma}_2 \times \bar{\Gamma}_1$ .
- Likewise, for a matrix and a spinor,  $\overline{(\Gamma \times \Psi)} = \bar{\Psi} \times \bar{\Gamma}$ .
- The Dirac conjugate of a complex number is its complex conjugate,  $\overline{(c \times \mathbf{1})} = c^* \times \mathbf{1}$ .
- For any two spinors  $\Psi_1$  and  $\Psi_2$  and any matrix  $\Gamma$ ,  $\bar{\Psi}_1 \bar{\Gamma} \Psi_2 = (\bar{\Psi}_2 \Gamma \Psi_1)^*$ .
  - The Dirac spinor fields are fermionic, so they anticommute with each other, even in the classical limit. Nevertheless,  $(\Psi_\alpha^\dagger \Psi_\beta)^\dagger = +\Psi_\beta^\dagger \Psi_\alpha$ , and therefore for any matrix  $\Gamma$ ,  $\bar{\Psi}_1 \bar{\Gamma} \Psi_2 = +(\bar{\Psi}_2 \Gamma \Psi_1)^*$ .

The point of the Dirac conjugation (52) is that it works similarly for all four Dirac matrices  $\gamma^\mu$ ,

$$\overline{\gamma^\mu} = +\gamma^\mu. \quad (53)$$

Proof: For  $\mu = 0$ , the  $\gamma^0$  is hermitian, hence

$$\overline{\gamma^0} = \gamma^0 (\gamma^0)^\dagger \gamma^0 = +\gamma^0 \gamma^0 \gamma^0 = +\gamma^0. \quad (54)$$

For  $\mu = i = 1, 2, 3$ , the  $\gamma^i$  are anti-hermitian and also anticommute with the  $\gamma^0$ , hence

$$\overline{\gamma^i} = \gamma^0 (\gamma^i)^\dagger \gamma^0 = -\gamma^0 \gamma^i \gamma^0 = +\gamma^0 \gamma^0 \gamma^i = +\gamma^i. \quad (55)$$

**Corollary:** *The Dirac conjugate of the matrix*

$$M_D(L) = \exp\left(-\frac{i}{2}\Theta_{\mu\nu}S^{\mu\nu}\right) \quad (20)$$

*representing any continuous Lorentz symmetry  $L = \exp(\Theta)$  is the inverse matrix*

$$\overline{M}_D(L) = M_D^{-1}(L) = \exp\left(+\frac{i}{2}\Theta_{\mu\nu}S^{\mu\nu}\right). \quad (56)$$

Proof: Let

$$X = -\frac{i}{2}\Theta_{\mu\nu}S^{\mu\nu} = +\frac{1}{8}\Theta_{\mu\nu}[\gamma^\mu, \gamma^\nu] = +\frac{1}{4}\Theta_{\mu\nu}\gamma^\mu\gamma^\nu \quad (57)$$

for some real antisymmetric Lorentz parameters  $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$ . The Dirac conjugate of the  $X$  matrix is

$$\overline{X} = \overline{\frac{1}{4}\Theta_{\mu\nu}\gamma^\mu\gamma^\nu} = \frac{1}{4}\Theta_{\mu\nu}^*\overline{\gamma^\nu\gamma^\mu} = \frac{1}{4}\Theta_{\mu\nu}\gamma^\nu\gamma^\mu = \frac{1}{4}\Theta_{\nu\mu}\gamma^\mu\gamma^\nu = -\frac{1}{4}\Theta_{\mu\nu}\gamma^\mu\gamma^\nu = -X. \quad (58)$$

Consequently,

$$\overline{X^2} = \overline{X} \times \overline{X} = +X^2, \quad \overline{X^3} = \overline{X} \times \overline{X^2} = \overline{X^2} \times \overline{X} = -X^3, \quad \dots, \quad \overline{X^n} = (-X)^n, \quad (59)$$

and hence

$$\overline{\exp(X)} = \sum_n \frac{1}{n!} \overline{X^n} = \sum_n \frac{1}{n!} (-X)^n = \exp(-X). \quad (60)$$

In light of eq. (57), this means

$$\overline{\exp\left(-\frac{i}{2}\Theta_{\mu\nu}S^{\mu\nu}\right)} = \exp\left(+\frac{i}{2}\Theta_{\mu\nu}S^{\mu\nu}\right), \quad (61)$$

that is,

$$\overline{M}_D(L) = M_D^{-1}(L). \quad (62)$$

*Quod erat demonstrandum.*

### Back to the Dirac Lagrangian:

Thanks to the theorem (62), the conjugate field  $\bar{\Psi}(x)$  in the Lagrangian (47) is simply the Dirac conjugate of the Dirac spinor field  $\Psi(x)$ ,

$$\bar{\Psi}(x) = \Psi^\dagger(x) \times \gamma^0, \quad (63)$$

which transforms under Lorentz symmetries as

$$\bar{\Psi}'(x') = \overline{\Psi'(x')} = \overline{M_D(L) \times \Psi(x)} = \bar{\Psi}(x) \times \overline{M_D(x)} = \bar{\Psi}(x) \times M_D^{-1}(L). \quad (64)$$

Consequently, the Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi} \times (i\gamma^\mu \partial_\mu - m)\Psi = \Psi^\dagger \gamma^0 \times (i\gamma^\mu \partial_\mu - m)\Psi \quad (47)$$

is a Lorentz scalar and the action is Lorentz invariant.

## Hamiltonian for the Dirac Field

Canonical quantization of the Dirac spinor field  $\Psi(x)$  — just like any other field — begins with the classical Hamiltonian formalism. Let's start with the canonical conjugate fields,

$$\Pi_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi_\alpha)} = (i\bar{\Psi}\gamma^0)_\alpha = i\Psi_\alpha^\dagger \quad (65)$$

— the canonical conjugate to the Dirac spinor field  $\Psi(x)$  is simply its hermitian conjugate  $\Psi^\dagger(x)$ . This is similar to what we had for the non-relativistic field, and it happens for the same reason — the Lagrangian which is linear in the time derivative.

In the non-relativistic field theory, the conjugacy relation (65) in the classical theory lead to the equal-time commutation relations in the quantum theory,

$$[\hat{\psi}(\mathbf{x}, t), \hat{\psi}(\mathbf{y}, t)] = 0, \quad [\hat{\psi}^\dagger(\mathbf{x}, t), \hat{\psi}^\dagger(\mathbf{y}, t)] = 0, \quad [\hat{\psi}(\mathbf{x}, t), \hat{\psi}^\dagger(\mathbf{y}, t)] = \delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (66)$$

However, the Dirac spinor field describes spin =  $\frac{1}{2}$  particles — like electrons, protons, or neutrons — which are fermions rather than bosons. So instead of the commutations rela-

tions (66), the spinor fields obey the *equal-time anti-commutation relations*

$$\begin{aligned}
\{\hat{\Psi}_\alpha(\mathbf{x}, t), \hat{\Psi}_\beta(\mathbf{y}, t)\} &= 0, \\
\{\hat{\Psi}_\alpha^\dagger(\mathbf{x}, t), \hat{\Psi}_\beta^\dagger(\mathbf{y}, t)\} &= 0, \\
\{\hat{\Psi}_\alpha(\mathbf{x}, t), \hat{\Psi}_\beta^\dagger(\mathbf{y}, t)\} &= \delta_{\alpha\beta} \delta^{(3)}(\mathbf{x} - \mathbf{y}).
\end{aligned} \tag{67}$$

Next, the classical Hamiltonian obtains as

$$\begin{aligned}
H &= \int d^3\mathbf{x} \mathcal{H}(\mathbf{x}), \\
\mathcal{H} &= i\Psi^\dagger \partial_0 \Psi - \mathcal{L} \\
&= i\Psi^\dagger \partial_0 \Psi - \Psi^\dagger \gamma^0 (i\gamma^0 \partial_0 + i\vec{\gamma} \cdot \nabla - m) \Psi \\
&= \Psi^\dagger (-i\gamma^0 \vec{\gamma} \cdot \nabla + \gamma^0 m) \Psi
\end{aligned} \tag{68}$$

where the terms involving the time derivative  $\partial_0$  cancel out. Consequently, the Hamiltonian operator of the quantum field theory is

$$\hat{H} = \int d^3\mathbf{x} \hat{\Psi}^\dagger(\mathbf{x}) (-i\gamma^0 \vec{\gamma} \cdot \nabla + \gamma^0 m) \hat{\Psi}(\mathbf{x}). \tag{69}$$

Note that the derivative operator  $(-i\gamma^0 \vec{\gamma} \cdot \nabla + \gamma^0 m)$  in this formula is precisely the 1-particle Dirac Hamiltonian (28) in the coordinate basis. This is very similar to what we had for the quantum non-relativistic fields,

$$\hat{H} = \int d^3\mathbf{x} \hat{\psi}^\dagger(\mathbf{x}) \left( \frac{-1}{2M} \nabla^2 + V(\mathbf{x}) \right) \hat{\psi}(\mathbf{x}), \tag{70}$$

except for a different differential operator, Schrödinger instead of Dirac.

In the Heisenberg picture, the quantum Dirac field obeys the Dirac equation. To see how this works, we start with the Heisenberg equation

$$i \frac{\partial}{\partial t} \hat{\Psi}_\alpha(\mathbf{x}, t) = [\hat{\Psi}_\alpha(\mathbf{x}, t), \hat{H}] = \int d^3\mathbf{y} [\hat{\Psi}_\alpha(\mathbf{x}, t), \hat{\mathcal{H}}(\mathbf{y}, t)], \tag{71}$$

and then evaluate the last commutator using the anti-commutation relations (67) and the

Leibniz rules (11). Specifically, let's use the Leibniz rule

$$[A, BC] = \{A, B\}C - B\{A, C\} \quad (72)$$

for

$$\begin{aligned} A &= \hat{\Psi}_\alpha(\mathbf{x}, t), \\ B &= \hat{\Psi}_\beta^\dagger(\mathbf{y}, t), \\ C &= (-i\gamma^0\vec{\gamma} \cdot \nabla + \gamma^0 m)_{\beta\gamma} \hat{\Psi}_\gamma(\mathbf{y}, t), \end{aligned} \quad (73)$$

so that  $BC = \hat{\mathcal{H}}(\mathbf{y}, t)$ . For the  $A, B, C$  at hand,

$$\{A, B\} = \delta_{\alpha\beta} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (74)$$

while

$$\{A, C\} = (-i\gamma^0\vec{\gamma} \cdot \nabla_y + \gamma^0 m)_{\beta\gamma} \{\hat{\Psi}_\alpha(\mathbf{x}, t), \hat{\Psi}_\gamma(\mathbf{y}, t)\} = (\text{diff.op.}) \times 0 = 0. \quad (75)$$

Consequently

$$\begin{aligned} [\hat{\Psi}_\alpha(\mathbf{x}, t), \hat{\mathcal{H}}(\mathbf{y}, t)] &\equiv [A, BC] \\ &= \{A, B\} \times C - B \times \{A, C\} \\ &= \delta_{\alpha\beta} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \times (-i\gamma^0\vec{\gamma} \cdot \nabla + \gamma^0 m)_{\beta\gamma} \hat{\Psi}_\gamma(\mathbf{y}, t) - \Psi_\beta^\dagger(\mathbf{y}) \times 0 \\ &= \delta^{(3)}(\mathbf{x} - \mathbf{y}) \times (-i\gamma^0\vec{\gamma} \cdot \nabla + \gamma^0 m)_{\alpha\gamma} \hat{\Psi}_\gamma(\mathbf{y}, t), \end{aligned} \quad (76)$$

hence

$$\begin{aligned} [\hat{\Psi}_\alpha(\mathbf{x}, t), \hat{H}] &= \int d^3\mathbf{y} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \times (-i\gamma^0\vec{\gamma} \cdot \nabla + \gamma^0 m)_{\alpha\gamma} \hat{\Psi}_\gamma(\mathbf{y}, t) \\ &= (-i\gamma^0\vec{\gamma} \cdot \nabla + \gamma^0 m)_{\alpha\gamma} \hat{\Psi}_\gamma(\mathbf{x}, t), \end{aligned} \quad (77)$$

and therefore

$$i\partial_0 \hat{\Psi}(\mathbf{x}, t) = (-i\gamma^0\vec{\gamma} \cdot \nabla + \gamma^0 m) \hat{\Psi}(\mathbf{x}, t). \quad (78)$$

Or if you prefer,

$$(i\gamma^\mu \partial_\mu - m) \hat{\Psi}(x) = 0. \quad (79)$$