Glashow–Weinberg–Salam Theory

Glashow–Weinberg–Salam theory is a unified theory of weak and electromagnetic interactions. At its core is the $SU(2)_W \times U(1)_Y$ gauge theory spontaneously broken down to the $U(1)_{\rm EM}$. Out of 4 gauge fields W^a_μ ($a = 1,2,3$) and B_μ, one linear combination remains massless and gives rise to the electromagnetism, while 3 other linear combinations become massive and give rise to the weak interactions.

The key to the spontaneous breakdown of the electroweak gauge symmetry is the doublet of complex scalar fields H^α ($\alpha = 1,2$) called *the Higgs fields*. The $SU(2)_W \times U(1)_Y$ quantum numbers of these fields are $(\mathbf{2}, +\frac{1}{2})$; that is, they form a doublet of the $SU(2)_W$ and have the $U(1)_Y$ *hypercharge* $y = +\frac{1}{2}$. Thus,

$$D_\mu H^\alpha(x) \;=\; \partial_\mu H^\alpha + \frac{ig_2}{2} W^a_\mu(x) \bigl(\tau^a\bigr)^\alpha_{\ \beta} H^\beta(x) \;+\; \frac{ig_1}{2} B_\mu H^\alpha \tag{1}$$

where g_2 is the $SU(2)_W$ gauge coupling and g_1 is the $U(1)_Y$ gauge coupling.

The gauge fields W^a_μ and B_μ and the Higgs fields H_α are the only bosonic fields of the GWS theory. There are also 24 fermionic fields describing the quarks and the leptons — I have a separate set of notes about them — but let's take care of the bosons first. The bosonic part of the theory's Lagrangian is

$$\mathcal{L} \;=\; -\tfrac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} \;-\; \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} \;+\; D_\mu H^\dagger D^\mu H \;-\; \frac{\lambda}{2}\left(H^\dagger H \;-\; \frac{v^2}{2}\right)^2 \;+\; \text{fermionic terms} \tag{2}$$

where

$$\begin{aligned} B_{\mu\nu} \;&=\; \partial_\mu B_\nu \;-\; \partial_\nu B_\mu \,, \\ W^a_{\mu\nu} \;&=\; \partial_\mu W^a_\nu \;-\; \partial_\nu W^a_\mu \;-\; g_2 \epsilon^{abc} W^b_\mu W^c_\nu \,, \\ H \;&=\; \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad H^\dagger \;=\; \bigl(H_1^*, H_2^*\bigr), \end{aligned} \tag{3}$$

and $D_\mu H$, $D_\mu H^\dagger$ are row/column vector forms of $D_\mu H_\alpha$ and $D_\mu H^*_\alpha$ from eq. (1). The scalar potential $V = \frac{\lambda}{2}\left(H^\dagger H - \frac{v^2}{2}\right)^2$ has a local maximum rather than a minimum at $H = 0$, while its minima form a spherical shell $H^\dagger H = \frac{v^2}{2}$ in the scalar field space $\mathbf{C}^2 = \mathbf{R}^4$. All such minima are related to each other by gauge symmetries, so without loss of generality we assume the Higgs fields have Vacuum Expectation Values (VEVs)

$$\langle H \rangle \;=\; \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4}$$

Note that this expectation value breaks 3 out of 4 gauge symmetries of the theory, but one combination of the $U(1)_Y$ and a $U(1)$ subgroup of the $SU(2)_2$ remains unbroken. Indeed, the $U(1)_Y$ symmetry $\exp\bigl(i\Theta(x)\hat{Y}\bigr)$ acts on the Higgs fields as $H(x) \to \exp\bigl(iy\Theta(x)\bigr)H(x) = \exp\bigl(\frac{i}{2}\Theta(x)\bigr)H(x)$ since H has $y = +\frac{1}{2}$, while the $SU(2)$ symmetry $\exp\bigl(i\Theta(x)\hat{T}^3\bigr)$ — for the same $\Theta(x)$ — acts on the $SU(2)$ double H as $H(x) \to \exp\bigl(\frac{i}{2}\Theta(x)\tau^3\bigr)H(x)$. Combining the two symmetries, we have

$$H(x) \;\to\; \exp\bigl(\tfrac{i}{2}\Theta(x)\bigr)\exp\bigl(\tfrac{i}{2}\Theta(x)\tau^3\bigr)H(x) \;=\; \begin{pmatrix} e^{i\Theta(x)} & 0 \\ 0 & 1 \end{pmatrix} H(x), \tag{5}$$

which indeed leaves the vacuum expectation value (4) invariant. Thus, the $U(1)$ subgroup of the electroweak $SU(2)_W \times U(1)_Y$ generated by the operator

$$\hat{Q} \;=\; \hat{Y} \;+\; \hat{T}^3 \tag{6}$$

remains unbroken. Physically, this subgroup is the $U(1)_Q$ gauge symmetry of the electromagnetism and $\hat{Q}$ is the electric charge operator (or rather electric charge in units of e).

We shall see in a moment that one linear combination of the four $SU(2)_W \times U(1)_Y$ gauge fields corresponding to the $\hat{Q}$ generator remains massless while the other 3 combinations become massive via the Higgs mechanism. The same mechanism also eliminates 3 scalar fields, which becomes the longitudinal components of the 3 massive vector fields. Since the 2 complex Higgs fields are equivalent to 4 real scalars, we end up with $4 - 3 = 1$ physical scalar field $h(x)$; its quanta — called the *physical Higgs particles* — were experimentally discovered at the LHC in 2013.

The simplest way to see how this works is to fix the unitary gauge for the spontaneously broken symmetries. Note that any complex doublet $H(x)$ can be $SU(2)$–rotated to

$$H'(x) \;=\; U(x)H(x) \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \tilde{h}(x) \end{pmatrix} \tag{7}$$

for a *real* $\tilde{h}(x) \geq 0$. This gauge transform would be singular for $H(x) \approx 0$ but it is nice and smooth for $H(x)$ in the vicinity of the vacuum expectation value (4), so we may use it to fix the unitary gauge $H_1(x) \equiv 0$, $\mathrm{Im}(H_2(x)) \equiv 0$. Once we fix this gauge, we are left with a single real scalar field $\tilde{h}(x)$, which we may now shift by its VEV,

$$\tilde{h}(x) \;=\; v \;+\; h(x). \tag{8}$$

In terms of this shifted field,

$$H^\dagger H \,-\, \frac{v^2}{2} \;=\; \frac{(v+h)^2}{2} \,-\, \frac{v^2}{2} \;=\; vh \,+\, \tfrac{1}{2}h^2, \tag{9}$$

so the scalar potential becomes

$$V(h) \;=\; \frac{\lambda}{2}\left(H^\dagger H \,-\, \frac{v^2}{2}\right)^2 \;=\; \frac{\lambda}{2}\left(vh \,+\, \tfrac{1}{2}h^2\right)^2 \;=\; \frac{\lambda v^2}{2}\times h^2 \,+\, \frac{\lambda v}{2}\times h^3 \,+\, \frac{\lambda}{8}\times h^4 \tag{10}$$

with a positive mass$^2 = \lambda v^2 > 0$ for the physical Higgs field. Experimentally, $v = 247$ GeV while the physical Higgs mass is 125 GeV, which means $\lambda \approx 0.26$.

The mass terms for the vector fields emerge from the kinetic term $D_\mu H^\dagger D^\mu H$ for the Higgs doublets. Indeed, in the unitary gauge

$$D_\mu H \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{i}{2}g_2\big(W^1_\mu - iW^2_\mu\big)\tilde{h} \\ \partial_\mu\tilde{h} \,+\, \frac{i}{2}\big(g_1 B_\mu - g_2 W^3_\mu\big)\tilde{h} \end{pmatrix} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{i}{2}g_2\big(W^1_\mu - iW^2_\mu\big)(v+h) \\ \partial_\mu h \,+\, \frac{i}{2}\big(g_1 B_\mu - g_2 W^3_\mu\big)(v+h) \end{pmatrix} \tag{11}$$

and hence

$$\begin{aligned} D_\mu H^\dagger D^\mu H \;&=\; \tfrac{1}{2}\Big|\partial_\mu h \,+\, \tfrac{i}{2}\big(g_1 B_\mu - g_2 W^3_\mu\big)(v+h)\Big|^2 \,+\, \tfrac{1}{2}\Big|\tfrac{i}{2}g_2\big(W^1_\mu - iW^2_\mu\big)(v+h)\Big|^2 \\ &=\; \tfrac{1}{2}(\partial_\mu h)^2 \,+\, \frac{(v+h)^2}{8}\Big(g_1 B_\mu - g_2 W^3_\mu\Big)^2 \,+\, \frac{g_2^2(v+h)^2}{8}\Big(\big(W^1_\mu\big)^2 + \big(W^2_\mu\big)^2\Big). \end{aligned} \tag{12}$$

The first term on the last line here is the kinetic term for the physical Higgs field while the rest are the mass terms for the vector fields and also their interactions with the physical

Higgs field $h(x)$. In particular, the vector mass terms obtain from truncating the $(v+h(x))^2$ factors to simply v^2, thus

$$\mathcal{L}^{\text{vector}}_{\text{masses}} \;=\; \frac{g_2^2 v^2}{8}\times\left(\left(W^1_\mu\right)^2+\left(W^2_\mu\right)^2\right) \;+\; \frac{v^2}{8}\times\left(g_1 B_\mu - g_2 W^3_\mu\right)^2. \tag{13}$$

In particular, the W^1_μ and W^2_μ vector fields have masses

$$M_W^2 \;=\; \frac{g_2^2 v^2}{4} \quad\Longrightarrow\quad M_W \;=\; \frac{g_2 v}{2}\,, \tag{14}$$

while the W^3_μ and B_μ vector fields have a 2×2 mass matrix

$$M^2 \;=\; \frac{v^2}{4}\begin{pmatrix} g_2^2 & -g_2 g_1 \\ -g_2 g_1 & g_1^2 \end{pmatrix}. \tag{15}$$

This matrix has eigenvalues

$$M_Z^2 \;=\; \frac{(g_2^2+g_1^2)v^2}{4} \quad\text{and}\quad M_A^2 \;=\; 0 \tag{16}$$

— as promised, there is one massless vector field — while the mass eigenstates correspond to the vector fields

$$\begin{aligned} \text{massive } Z_\mu(x) \;&=\; \cos\theta\times W^3_\mu(x) \;-\; \sin\theta\times B_\mu(x), \\ \text{massless } A_\mu(x) \;&=\; \sin\theta\times W^3_\mu(x) \;+\; \cos\theta\times B_\mu(x), \end{aligned} \tag{17}$$

where

$$\theta \;=\; \arctan\frac{g_1}{g_2} \tag{18}$$

is the *weak mixing angle* or the Weinberg angle; experimentally, $\sin^2\theta\approx 0.23$.

Physically, the $A_\mu(x)$ is the EM field whose quanta are massless photons, the $Z_\mu(x)$ is the neutral weak field whose quanta are Z^0 particles of mass $M_Z\approx 91$ GeV, and the $W^{1,2}_\mu(x)$

— or rather their linear combinations

$$W^+_\mu(x) \;=\; \frac{W^1_\mu(x)\,+\,iW^2_\mu(x)}{\sqrt{2}} \quad\text{and}\quad W^-_\mu(x) \;=\; \frac{W^1_\mu(x)\,-\,iW^2_\mu(x)}{\sqrt{2}} \tag{19}$$

— are the charged weak fields (electric charges $q=\pm1$) whose quanta are the W^+ and W^- particles of mass $M_W\approx 80$ GeV. The experimentally found mass ratio between the $W^\pm$ and Z^0 particles gives us the value of the weak mixing angle:

$$\frac{M_W^2}{M_Z^2} \;=\; \frac{g_2^2}{g_2^2+g_1^2} \;=\; \frac{1}{1+\tan^2\theta} \;=\; \cos^2\theta \quad\Longrightarrow\quad \cos^2\theta\approx 0.77 \;\Longrightarrow\; \sin^2\theta\approx 0.23. \tag{20}$$

*   *   *

Now lets find the currents to which the vector fields $W^\pm_\mu$, Z_μ, and A_μ couple and the strengths of those couplings. Of particular importance is the EM coupling strength e since it acts as the unit of the conventionally normalized electric charge, so we would like to relate it to the original $SU(2)_W\times U(1)_Y$ couplings g_2 and g_1. But the weak currents and couplings are also important.

Our starting point is the $SU(2)_W\times U(1)_Y$ symmetry currents J^Y_μ, J^{T1}_μ, J^{T2}_μ, J^{T3}_μ of the fermionic fields. Without going into the details of these currents — they are described in detail in my notes on quarks and leptons — we can say that the original gauge fields $B_\mu(x)$ and $W^a_\mu(x)$ couple to these currents according to

$$\mathcal{L}_{\rm net} \;\supset\; \mathcal{L}_{\rm current} \;=\; -g_2W^1_\mu\times J^\mu_{T1} \;-\; g_2W^2_\mu\times J^\mu_{T2} \;-\; g_2W^3_\mu\times J^\mu_{T3} \;-\; g_1B_\mu\times J^\mu_Y\,. \tag{21}$$

Now let's relate the original gauge fields to the vector fields of definite masses and electric charges. Inverting eqs. (19) and (17), we obtain

$$\begin{aligned} W^1_\mu \;&=\; \frac{1}{\sqrt{2}}\times W^-_\mu \;+\; \frac{1}{\sqrt{2}}\times W^+_\mu\,,\\ W^2_\mu \;&=\; \frac{i}{\sqrt{2}}\times W^-_\mu \;-\; \frac{i}{\sqrt{2}}\times W^+_\mu\,,\\ W^3_\mu \;&=\; \cos\theta\times Z_\mu \;+\; \sin\theta\times A_\mu\,,\\ B_\mu \;&=\; -\sin\theta\times Z_\mu \;+\; \cos\theta\times A_\mu\,. \end{aligned} \tag{22}$$

Plugging these formulae into eq. (21) and re-arranging the terms, we find

$$\begin{aligned}\mathcal{L}_{\text{current}} \;&=\; -\frac{g_2}{\sqrt{2}}\,W^-_\mu \times \bigl(J^\mu_{T1} - iJ^\mu_{T2}\bigr) \;-\; \frac{g_2}{\sqrt{2}}\,W^+_\mu \times \bigl(J^\mu_{T1} + iJ^\mu_{T2}\bigr)\\ &\quad-\; Z_\mu \times \bigl(g_2\cos\theta J^\mu_{T3} \,-\, g_1\sin\theta\, J^\mu_Y\bigr) \;-\; A_\mu \times \bigl(g_2\sin\theta J^\mu_{T3} \,+\, g_1\cos\theta\, J^\mu_Y\bigr),\end{aligned} \tag{23}$$

or in other words

$$\mathcal{L}_{\text{current}} \;=\; -\frac{g_2}{\sqrt{2}}\Bigl(W^+_\mu \times J^{-\mu} \,+\, W^-_\mu \times J^{+\mu}\Bigr) \;-\; \tilde{g} Z_\mu \times J^\mu_Z \;-\; eA_\mu \times J^\mu_{\text{EM}} \tag{24}$$

where

$$J^{+\mu} \;=\; J^\mu_{T1} \,-\, iJ^\mu_{T2}\,, \qquad J^{-\mu} \;=\; J^\mu_{T1} \,+\, iJ^\mu_{T2}\,, \tag{25}$$

are the charged weak currents,

$$\tilde{g} \times J^\mu_Z \;=\; g_2\cos\theta J^\mu_{T3} \;-\; g_1\sin\theta\, J^\mu_Y \tag{26}$$

is the neutral weak current (times the neutral weak coupling constant), and

$$e \times J^\mu_{\text{EM}} \;=\; g_2\sin\theta J^\mu_{T3} \;+\; g_1\cos\theta\, J^\mu_Y \tag{27}$$

is the (conventionally normalized) electric current. Note that on the right hand side of this formula $g_1\cos\theta = g_2\sin\theta$ because of the way the weak mixing angle θ is related to the gauge couplings, $\tan\theta = g_1/g_2$, *cf.* eq. (18). Consequently, we may identify

$$e \;=\; g_2\sin\theta \;=\; g_1\cos\theta \quad\Longrightarrow\quad \frac{1}{e^2} \;=\; \frac{1}{g_2^2}\left(\frac{1}{\sin^2\theta} \;=\; 1+\frac{1}{\tan^2\theta}\right) \;=\; \frac{1}{g_2^2} \;+\; \frac{1}{g_1^2} \tag{28}$$

and

$$J^\mu_{\text{EM}} \;=\; J^\mu_{T3} \;+\; J^\mu_Y\,. \tag{29}$$

Note that this current does not depend on the gauge couplings or θ; instead, it's the current of the electric charge operator $\hat{Q} = \hat{T}^3 + \hat{Y}$ which is the generator of the *unbroken* $U(1)_{\text{EM}}$ gauge symmetry. Naturally, the EM field $A_\mu(x)$ — which is the gauge field of that $U(1)_{\text{EM}}$ — should couple to precisely this symmetry current.

On the other hand, the Z_μ is the gauge field of a spontaneously broken symmetry, so the specific combination of the symmetry currents that couples to the Z_μ depends on the weak mixing angle. Indeed, the coefficients of the two terms on the RHS of eq. (26) are quite different and their ratio depends on g_1/g_2; specifically,

$$\begin{aligned}
g_2 \times \cos\theta \;&=\; \frac{g_2^2}{\sqrt{g_2^2+g_1^2}} \;=\; \sqrt{g_2^2+g_1^2} \times \cos^2\theta, \\
g_1 \times \sin\theta \;&=\; \frac{g_1^2}{\sqrt{g_2^2+g_1^2}} \;=\; \sqrt{g_2^2+g_1^2} \times \sin^2\theta, \\
\frac{g_1 \times \sin\theta}{g_2 \times \cos\theta} \;&=\; \tan^2\theta.
\end{aligned} \tag{30}$$

Consequently, we may identify

$$\tilde{g} \;=\; \sqrt{g_2^2+g_1^2} \;=\; \frac{g_2}{\cos\theta} \;=\; \frac{g_1}{\sin\theta} \;=\; \frac{e}{\sin\theta\cos\theta} \tag{31}$$

and then the neutral weak current becomes

$$\begin{aligned}
J_Z^\mu \;&=\; \cos^2\theta \times J_{T3}^\mu \;-\; \sin^2\theta \times J_Y^\mu \\
&=\; J_{T3}^\mu \;-\; \sin^2\theta \times \left(J_{T3}^\mu \;+\; J_Y^\mu\right) \\
&=\; J_{T3}^\mu \;-\; \sin^2\theta \times J_{EM}^\mu\,.
\end{aligned} \tag{32}$$

Note that the weak couplings g_2 and $\tilde{g}$ are larger than the EM coupling e. Consequently, at high energies much larger than the masses of W and Z particles, the weak interactions are not weak at all — they are stronger them the EM interactions. But at low energies, the β-decays and other processes mediated by the *virtual* $W^\pm$ or Z^0 involve the propagators

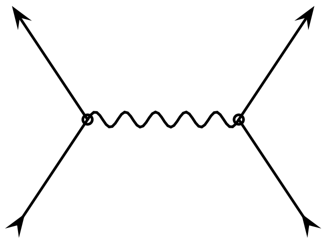
$$\begin{aligned}
W^\pm \text{ propagator } \;&\sim\; \frac{1}{q^2-M_W^2} \;\approx\; \frac{-1}{M_W^2}\,, \\
Z^0 \text{ propagator } \;&\sim\; \frac{1}{q^2-M_Z^2} \;\approx\; \frac{-1}{M_Z^2}\,,
\end{aligned} \tag{33}$$

so the overall weak amplitudes are

$$\mathcal{M} \;\sim\; \frac{g_2^2 E^2}{M_W^2} \quad\text{or}\quad \mathcal{M} \;\sim\; \frac{\tilde{g}^2 E^2}{M_Z^2}\,. \tag{34}$$

It's not the couplings, it's the small E^2/M_W^2 or E^2/M_Z^2 factors which make the weak interactions weak at low energies!

Indeed, consider a low-energy decay or scattering process mediated by a virtual $W^{\pm}$:



(35)

The left fermionic line — the vertex and the external line factors — stems from the J_μ^- current of the two fermions involved and its coupling to the $W^{+\mu}$; the general form of this line is $(ig_2/\sqrt{2}) \times J_\mu^-$. Likewise, the general form of the right line is $(ig_2/\sqrt{2}) \times J_\nu^+$. Multiplying these two factors by the W propagator, we have

$$i\mathcal{M} \;=\; \frac{ig_2}{\sqrt{2}}\,J_\mu^- \times \frac{ig_2}{\sqrt{2}}\,J_\nu^+ \times \frac{i}{q^2 - M_W^2}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_W^2}\right). \tag{36}$$

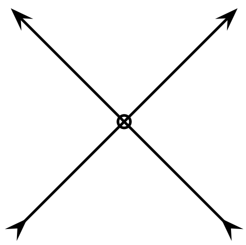
At low energies $|q^2| \ll M_W^2$, hence

$$\frac{i}{q^2 - M_W^2}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_W^2}\right) \;\approx\; \frac{ig^{\mu\nu}}{M_W^2} \tag{37}$$

and therefore

$$i\mathcal{M} \;\approx\; \frac{ig_2^2}{2M_W^2} \times J_\mu^- J^{+\mu}\,. \tag{38}$$

Diagrammatically, this approximation corresponds to



(39)

where the 4–fermion vertex stems from the effective low-energy Lagrangian

$$\mathcal{L}_{\text{eff}} \;=\; -\frac{g_2^2}{2M_W^2} \times J_\mu^- J^{+\mu}\,. \tag{40}$$

Historically, this effective Lagrangian was written down by Enrico Fermi back in 1933, long before the modern Glashow–Weinberg–Salam theory or the experimental discovery of the W particle. To be precise, Fermi wrote

$$\mathcal{L} = -\frac{G}{\sqrt{2}} \times J_\mu^- J^{+\mu} \quad \text{for} \quad J_\mu^\pm = \sum_{\substack{\text{appropriate}\\ \text{species}\\ \text{of fermions}}} \overline{\Psi}\gamma_\mu \Psi \tag{41}$$

but the discovery of parity violation in 1957 modified Fermi's original theory to

$$J_\mu^\pm = \sum_{\substack{\text{appropriate}\\ \text{species}\\ \text{of fermions}}} \overline{\Psi}\gamma_\mu (1-\gamma^5)\Psi. \tag{42}$$

As we shall see in my notes on the fermions in the GWS theory, this is indeed the correct form of the charged weak currents (apart from the overall factor of 2).

The overall constant G in the Fermi Lagrangian (41) nowadays is called the *Fermi constant*; its experimental value (in $c = \hbar = 1$ units) is $G \approx 1.166\ \text{GeV}^{-2}$. In the Glashow–Weinberg–Salam theory, this constant arises as

$$4 \times \frac{G}{\sqrt{2}} = \frac{g_2^2}{2M_W^2} = \frac{g_2^2}{\frac{1}{2}g_2^2 v^2} = \frac{2}{v^2}\,, \tag{43}$$

so the experimental value of G translates to the Higgs VEV $v \approx 247$ GeV.

Finally, besides the charged massive vectors $W^\pm$ coupled to the charged weak currents, the Glashow–Weinberg–Salam theory also has a neutral massive vectors Z^0 coupled to the neutral weak current J_Z^μ. Consequently, the effective low-energy theory of weak interaction has form

$$\mathcal{L}_{\text{eff}} = -\frac{g_2^2}{2M_W^2} \times J_\mu^- J^{+\mu} - \frac{\tilde{g}^2}{2M_Z^2} \times J_{Z\mu} J_Z^\mu\,, \tag{44}$$

or in terms of the Fermi constant G,

$$\mathcal{L}_{\text{eff}} = -2\sqrt{2}G\Big(J_\mu^- J^{+\mu} + \rho (J_Z^\mu)^2\Big) \tag{45}$$

where

$$\rho = \frac{\tilde{g}^2}{M_Z^2} \Bigg/ \frac{g_2^2}{M_W^2}\,. \tag{46}$$

Experimentally, the neutral–current weak interactions were discovered in 1973, and eventually their relative strength ρ was measured with high precision to be $\rho = 1$, in perfect agreement with the Glashow–Weinberg–Salam theory where

$$\frac{g_2}{\tilde{g}} \; = \; \cos\theta_w \; = \; \frac{M_W}{M_Z} \quad \Longrightarrow \quad \rho \; = \; 1. \tag{47}$$