Lorentz Symmetry of Particles and Fields

The *continuous* Lorentz symmetry group $SO^+(3,1)$ comprises Lorentz boosts in all directions, space rotations around all axes, and all combinations of rotations and boosts. The generators of this group form an antisymmetric tensor $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$, similar to the $\hat{J}^{ab} = -\hat{J}^{ba}$ generators of the $SO(N)$ symmetries. The commutation relations between the Lorentz generators follow from

$$\hat{J}^{\mu\nu}_{\text{orbital}} = ix^\mu\partial^\nu - ix^\nu\partial^\mu, \tag{1}$$

hence

$$\left[\hat{J}^{\kappa\lambda}, \hat{J}^{\mu\nu}\right] = ig^{\lambda\mu}\hat{J}^{\kappa\nu} - ig^{\kappa\mu}\hat{J}^{\lambda\nu} - ig^{\lambda\nu}\hat{J}^{\kappa\mu} + ig^{\kappa\nu}\hat{J}^{\lambda\mu}. \tag{2}$$

In 3D terms, the $\hat{J}^{ij} = \epsilon^{ijk}\hat{J}^k$ are the angular momenta which generate the 3–space rotations, while the $\hat{J}^{0i} = -\hat{J}^{i0} = \hat{K}^i$ generate the Lorentz boosts. The commutation relations (2) in terms of the $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ components become

$$\left[\hat{J}^i, \hat{J}^j\right] = i\epsilon^{ijk}\hat{J}^k, \quad \left[\hat{J}^i, \hat{K}^j\right] = i\epsilon^{ijk}\hat{K}^k, \quad \left[\hat{K}^i, \hat{K}^j\right] = -i\epsilon^{ijk}\hat{J}^k. \tag{3}$$

(The proof is part of problem#1 of homework#6.) Note the red minus sign in the commutator of the two boost generators; for similar generators of the compact $SO(4)$ group this sign would be plus. Consequently the quadratic Casimir⋆ of the Lorentz algebra

$$C_2 = \tfrac{1}{2}\hat{J}_{\mu\nu}\hat{J}^{\mu\nu} = \hat{\mathbf{J}}^2 - \hat{\mathbf{K}}^2 \tag{4}$$

is *not* positive definite, unlike the $SO(4)$ Casimir $\hat{\mathbf{J}}^2 + \hat{\mathbf{K}}^2$. A mixed-signature but non-degenerate Casimir like (4) means that the corresponding Lie group is simple but non-compact, unlike a simple and compact $SO(4)$ group. And while simple compact groups have plenty of finite unitary multiplets — in fact, every finite representation of such a group is unitary, — for the simple non-compact groups like Lorentz $SO^+(3,1)$ it's the other way around:

⋆ The quadratic Casimir of a Lie algebra is a quadratic combinations $C_2 = g_{ab}T^aT^b$ of the generators which commutes with the whole algebra, $[C_2, T^c] = 0\ \forall T^c$. For example, the quadratic Casimir of the $SO(3)$ algebra is $C_2 = \mathbf{J}^2$.

⋆ There are finite but non-unitary representations like the vector **4** or the tensors.

⋆ There are unitary but infinite representations like the mass shell.

• But there are no finite unitary representations except the trivial singlet.

In quantum field theory, the fields form finite but non-unitary multiplets like $A^\mu(x)$ or $F^{\mu\nu}(x)$, while the particle states form unitary but infinite multiplets.

Particle Representations of the Lorentz Group

Let's start with the spinless particles which do not have any polarization quantum numbers, so their quantum states $|p\rangle$ can be labeled just by the 3–momentum $\mathbf{p}$, or equivalently by the energy-momentum 4–vector p^μ which lie on the mass shell $p^0 = +\sqrt{m^2+\mathbf{p}^2}$. Under a Lorentz symmetry, the momentum transforms as a 4–vector, $p \to p' = Lp$, or in components $p'^\mu = L^\mu{}_\nu p^\nu$, hence in the Hilbert space of the one-particle states

$$\widehat{\mathcal{D}}(L)\,|p\rangle \;=\; |Lp\rangle \tag{5}$$

for some *unitary* operator $\widehat{\mathcal{D}}(L)$. Eq. (5) is the prototypical particle representation of the Lorentz symmetry: It is unitary, but the members form a continuous family — the mass shell — rather than a finite set.

Now let's promote the particle representation (5) to the Fock space of arbitrary number of spinless particles. The vacuum state is Lorentz-invariant, thus $\widehat{\mathcal{D}}(L)\,|\text{vac}\rangle = |\text{vac}\rangle$ for any $L \in SO^+(3,1)$. At the same time, the one-particle states $|p\rangle = \hat{a}^\dagger_p\,|\text{vac}\rangle$ transform according to eq. (5), hence

$$\begin{array}{ccc} \widehat{\mathcal{D}}(L)\hat{a}^\dagger_p\,|\text{vac}\rangle & = & \hat{a}^\dagger_{Lp}\,|\text{vac}\rangle \\ \| & & \\ \widehat{\mathcal{D}}(L)\hat{a}^\dagger_p\widehat{\mathcal{D}}^\dagger(L)\,|\text{vac}\rangle & & \end{array} \tag{6}$$

and therefore

$$\widehat{\mathcal{D}}(L)\hat{a}^\dagger_p\widehat{\mathcal{D}}^\dagger(L) \;=\; \hat{a}^\dagger_{Lp}\,. \tag{7}$$

Consequently, for any multi-particle state

$$|p_1,p_2,\ldots,p_n\rangle \;=\; \hat{a}^\dagger_{p_1}\hat{a}^\dagger_{p_2}\cdots\hat{a}^\dagger_{p_n}\,|\text{vac}\rangle \tag{8}$$

we have

$$\begin{aligned}
\widehat{\mathcal{D}}(L)\left|p_1,p_2,\ldots,p_n\right\rangle &= \widehat{\mathcal{D}}(L)\hat{a}^\dagger_{p_1}\hat{a}^\dagger_{p_2}\cdots\hat{a}^\dagger_{p_n}\left|\text{vac}\right\rangle \\
&= \widehat{\mathcal{D}}(L)\hat{a}^\dagger_{p_1}\widehat{\mathcal{D}}^\dagger(L)\times\widehat{\mathcal{D}}(L)\hat{a}^\dagger_{p_2}\widehat{\mathcal{D}}^\dagger(L)\times\cdots \\
&\qquad\qquad \cdots\times\widehat{\mathcal{D}}(L)\hat{a}^\dagger_{p_n}\widehat{\mathcal{D}}^\dagger(L)\times\widehat{\mathcal{D}}(L)\left|\text{vac}\right\rangle \\
&= \hat{a}^\dagger_{Lp_1}\hat{a}^\dagger_{Lp_2}\cdots\hat{a}^\dagger_{Lp_n}\left|\text{vac}\right\rangle \\
&= \left|Lp_1,Lp_2,\ldots,Lp_n\right\rangle.
\end{aligned} \tag{9}$$

In other words, we have simultaneous Lorentz transform of every particle, $p_i \to Lp_i$.

⋆ ⋆ ⋆

Now consider particles with several spin or polarization states s. For such particles, the Hilbert space of one-particle states spans $|p,s\rangle$ where the momentum p^μ spans the mass shell, while s spans some *finite set* of spin or polarization states. The Lorentz symmetries act on such states as

$$\widehat{\mathcal{D}}(L)\left|p,s\right\rangle = \sum_{s'} C_{ss'}(L,p)\left|Lp,s'\right\rangle \tag{10}$$

where $C_{ss'}(L,p)$ is some matrix acting on the spin/polarization states. Eq. (10) is a more general kind of particle representation of the Lorentz group then (5), and we shall classify all such particle representations in a moment. But first, let's promote the Lorentz transform (10) to the Fock space of the multi-particle states. Similar to the spinless particle case, we have Lorentz-invariant vacuum state $|\text{vac}\rangle$ while the one-particle states $|p,s\rangle = \hat{a}^\dagger_{p,s}\left|\text{vac}\right\rangle$ transform according to eq. (10), hence

$$\widehat{\mathcal{D}}(L)\hat{a}^\dagger_{p,s}\widehat{\mathcal{D}}^\dagger(L) = \sum_{s'} C_{s,s'}(L,p)\times\hat{a}^\dagger_{Lp,s'} \tag{11}$$

and therefore

$$\widehat{\mathcal{D}}(L)\left|(p_1,s_1),\ldots,(p_n,s_n)\right\rangle = \sum_{s'_1,\ldots,s'_n} C_{s_1,s'_1}(L,p_1)\cdots C_{s_n,s'_n}(L,p_n)\times\left|(Lp_1,s'_1),\ldots,(Lp_n,s'_n)\right\rangle. \tag{12}$$

In other words, we have simultaneous Lorentz transform of all the particles according to eq. (10). Thus, given the one-particle representation (10) of the Lorentz group, all the multi-particle representations follow according to eq. (12).

This brings us our next task: Classifying all the one-particle representations (10), and in particular, classifying all the finite multiplets $\{s\}$ of the allowed spin/polarizations states s. The key to this classification is the *little group* $G(p)$ of the momentum 4–vector p^μ, that is, the subgroup of the Lorentz group which leaves p^μ invariant,

$$G(p) \;=\; \{L \in SO^+(3,1) \text{ such that } L^\mu_{\ \nu} p^\nu = p^\mu\}. \tag{13}$$

Indeed, let's pick any momentum p^μ on the mass shell, and any Lorentz transform in the little group of that momentum, $L \in G(p)$. Since L leaves p invariant, it acts only on the polarization/spin states of the particles with this momentum, thus

$$\text{for } L \in G(p): \quad \widehat{\mathcal{D}}(L)\,|p,s\rangle \;=\; \sum_{s'} C_{s,s'}(L,p)\,|p,s'\rangle \tag{14}$$

for the same $p' = Lp = p$. Moreover, since the operator $\widehat{\mathcal{D}}(L)$ is unitary — indeed, all the operators representing symmetries in the Hilbert space must be unitary, — the matrix $C_{s,s'}(L,p)$ must be a unitary matrix.

Now let's vary L but keep it within the little group $G(p)$ of some fixed momentum p^μ. Eq. (14) gives us a map from such $L \in G(p)$ to the unitary matrices $\|C_{s,s'}(L,p)\|$, so *this map is a finite unitary representation of the little group* $G(p)$. Therefore, to classify the finite multiplets of particle spin/polarization states, we need to classify the little groups $G(p)$ and their finite unitary representations. This was done back in 1926 by Eugine Wigner.

Here is the quick summary of the **Wigner Theorem:**

- ⋆ The massive particles have definite $SO(3)$ spins $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and their polarization states s come in multiplets of size $2j+1$.

- ⋆ The massless particles come in singlet multiplets $|p,\lambda\rangle$ of definite *helicity* λ, *i.e.*, the angular momentum component in the direction of the momentum $\mathbf{p}$. This helicity remains invariant under the continuous Lorentz transforms,

$$\forall L \in SO^+(3,1): \quad \widehat{\mathcal{D}}(L)\,|p,\lambda\rangle \;=\; |Lp,\lambda\rangle \times e^{i\,\text{phase}}. \tag{15}$$

- The tachyons — if they exist at all — do not have any spin or polarization quantum numbers. Their quantum states of definite momentum p come in trivial singlets $|p\rangle$.

Massive Particles

Let's start with a massive particle at rest, so its momentum is $p^\mu = (+m, \mathbf{0})$. Among the Lorentz symmetries, the 3–space rotations leave this momentum invariant while the boosts do not, so the little group is $G(p) = SO(3)$, the group of space rotations. The finite unitary representations of this group should be well known to the students of this class: they span states $|j, m\rangle$ for a fixed $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ while m runs from $-j$ to $+j$ by 1, thus $2j + 1$ states in a multiplet. In quantum field theory, each particle species has a definite spin j, so it has $2j + 1$ spin states labeled by m.

Now consider the same kind of a massive particle but moving at some sub-light velocity $\mathbf{v}$, $|\mathbf{v}| < 1$. Let $B_{\mathbf{v}}$ be the Lorentz by velocity $\mathbf{v}$, then $B_{\mathbf{v}}^{-1}$ puts the particle in its rest frame, $p' = B_{\mathbf{v}}^{-1} p = (m, \mathbf{0})$. Subsequently, any space rotation $R \in SO(3)$ preserve the rest momentum p', hence the original (moving) momentum p is preserved by $L = B_{\mathbf{v}} R B_{\mathbf{v}}^{-1}$,

$$R \times B_{\mathbf{v}}^{-1} p \;=\; B_{\mathbf{v}}^{-1} p \quad \Longrightarrow \quad B_{\mathbf{v}} R B_{\mathbf{v}}^{-1} p \;=\; p. \tag{16}$$

Thus, the little group of p comprises $L = B_{\mathbf{v}} R B_{\mathbf{v}}^{-1}$ for all space rotations R. Clearly, this little group is *isomorphic* to the rotation group, $G(p) \cong SO(3)$, so it has similar multiplet types to the $SO(3)$. Consequently, a massive particle of any velocity has a definite spin j and has $2j + 1$ spin states.

The generators of the little group $G(p)$ are the boosted angular momenta, $\tilde{J}^i = B_{\mathbf{v}} \hat{J}^i B_{\mathbf{v}}^{-1}$. Specifically, you shall see in your homework#6 (problem 2) that one of the generators is the *helicity* — the component $\hat{J}^{\|}$ of the angular momentum $\mathbf{J}$ in the direction of the velocity $\mathbf{v}$,

$$\hat{\lambda} \;=\; \tilde{J}^{\|} \;=\; \frac{\mathbf{v} \cdot \hat{\mathbf{J}}}{|v|}\,, \tag{17}$$

— while the other two generators are the two transverse (to $\mathbf{v}$) components of

$$\tilde{\mathbf{J}}^{\perp} \;=\; \gamma\big(\hat{\mathbf{J}} \,+\, \mathbf{v} \times \hat{\mathbf{K}}\big)^{\perp}. \tag{18}$$

Since $\hat{J}^{\|}$ is the only generator that's a pure angular momentum — as opposed to a mixture of an angular momentum and a boost generator, — it's convenient to organize the members of a spin multiplet (j) into eigenstates $|j,\lambda\rangle$ of the helicity $\hat{\lambda}$ rather than into eigenstates $|j,m\rangle$ of the $\tilde{J}^z$ generator. Thus, the spin multiplet for a massive particle comprises states $|j,\lambda\rangle$ for a fixed j and λ running over $2j+1$ values from $-j$ to $+j$.

The bottom line is, a particle representation of the Lorentz symmetry for a massive particle is parametrized by the mass or rather $m^2 > 0$ and spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ — where both m^2 and j have specific values for each particle species, — while the quantum states in this multiplet are $|p,\lambda\rangle$ where p spans the mass shell for mass m while λ runs from $-j$ to $+j$ by 1.

Massless Particles

A massless particle must move at the speed of light, hence $p^0 = +|\mathbf{p}|$ and there is no rest frame. So let's start with the example of the particle moving in $+\mathbf{z}$ direction, thus $p^\mu = (E, 0, 0, E)$. As you shall see in problem 2 of homework#6, the little group of this momentum is generated by

$$\hat{I}^x = \hat{J}^x - \hat{K}^y, \quad \hat{I}^y = \hat{J}^y + \hat{K}^x, \quad \text{and} \quad \hat{J}^z, \tag{19}$$

which obey the commutation relations

$$[\hat{J}^z, \hat{I}^x] = i\hat{I}^y, \quad [\hat{J}^z, \hat{I}^y] = -i\hat{I}^x, \quad [\hat{I}^x, \hat{I}^y] = 0. \tag{20}$$

These commutation relations are similar to the commutations relations of the $\hat{p}^x$, $\hat{p}^y$, and $\hat{J}^z$ — which together generate the translations and the rotations in the xy plane. Consequently, the little group of a lightlike momentum p^μ is isomorphic to the group if continuous isometries of a 2D plane, $G(p) \cong \mathrm{ISO}(2)$.

The ISO(2) group is non-compact but it is also non-simple. Consequently, it does have some non-trivial finite unitary representations. Specifically, they are non-trivial singlet states $|\lambda\rangle$ of definite helicity λ which are also annihilated by the $\hat{I}^{x,y}$ operators,

$$\hat{J}^z\,|\lambda\rangle = \lambda\,|\lambda\rangle\,, \quad \hat{I}^x\,|\lambda\rangle = 0, \quad \hat{I}^y\,|\lambda\rangle = 0. \tag{21}$$

For other directions of the massless particle's motion, the little group $G(p)$ is also isomorphic to ISO(2), and the generators of the $G(p)$ are the helicity $\hat{J}^{\|}$ and the two transverse (to the velocity) components $\hat{\mathbf{I}}^\perp$ of

$$\hat{\mathbf{I}} \;=\; \hat{\mathbf{J}} \;+\; \mathbf{v}\times\hat{\mathbf{K}}. \tag{22}$$

Again, the finite unitary multiplet of this group are singlet states $|\lambda\rangle$ of definite helicity which are also annihilated by the $\hat{\mathbf{I}}^\perp$ operators,

$$\hat{J}^{\parallel}\,|\lambda\rangle \;=\; \lambda\,|\lambda\rangle\,, \quad \hat{\mathbf{I}}^\perp\,|\lambda\rangle \;=\; 0. \tag{23}$$

Thus, the particle representations of the *continuous* Lorentz symmetry $SO^+(3,1)$ have definite helicities λ, with the states being $|p,\lambda\rangle$ where the momentum p^μ spans the $m^2=0$ mass shell while the helicity λ stays fixed.

Moreover, you shall see in problem#3 of homework#6 that for a massless particle with $p^2=0$, eqs. (23) can be summarized in a manifestly Lorentz-invariant way as

$$\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^{\gamma}\,|p,\lambda\rangle \;=\; \lambda p_\mu\,|p,\lambda\rangle\,. \tag{24}$$

This formula acts as an eigenvalue equation for the helicity λ, and its Lorentz invariance (under the continuous Lorentz symmetries only) means that the helicity itself is invariant under the continuous Lorentz transforms, thus

$$\widehat{\mathcal{D}}(L)\,|p,\lambda\rangle \;=\; |Lp,\text{same }\lambda\rangle\times e^{i\,\text{phase}}. \tag{25}$$

Thus, the continuous Lorentz symmetries do not change the helicities of massless particles. And that's why massless particles do not have multiplets of $2j+1$ different helicity states, instead each helicity eigenstate is a singlet multiplet all by itself. Indeed, the real-life observations confirm this rule:

- ⋆ The photons have helicities $\lambda=+1$ and $\lambda=-1$ but not $\lambda=0$. If the photons were massive, the Lorentz transformations would have turned the $|\lambda=\pm1\rangle$ states into combinations involving $|\lambda=0\rangle$, which would require them to have all 3 components of $j=1$ spin multiplet. But the real-life photons are strictly massive, so the mixing of different helicities never happen, and that's how the photons make do with the two transverse polarizations $\lambda=\pm1$ without having the longitudinal polarization $\lambda=0$.

* ⋆ The gravitons have helicities $\lambda = \pm 2$ but not $\lambda = \pm 1$ or $\lambda = 0$. Again, if the gravitons were massive they would need to have all 5 helicity states of the $j = 2$ multiplet, but the massless gravitons make do with just the $\lambda = \pm 2$ helicities without the other 3 values.

By the way, the reason photons and gravitons have two helicity values which differ by a sign — $\lambda = \pm 1$ for the photons and $\lambda = \pm 2$ for the gravitons, — is the *parity* symmetry P of the electromagnetism and gravity. The parity — which reverses the space coordinates but not the time, $P : (t, \mathbf{x}) \to (+t, -\mathbf{x})$, — flips the direction of the particle's velocity $\mathbf{v}$ but not of its angular momentum $\mathbf{J}$, so it changes the sign of the helicity $\lambda = \mathbf{v} \cdot \mathbf{J}/|v|$. Consequently, once we add the parity symmetry to the continuous Lorentz symmetries, it pairs up the massless $|p, +\lambda\rangle$ and $|p, -\lambda\rangle$ singlets of the $SO^+(3, 1)$ symmetry to a doublet of the $O^+(3, 1)$. (Except the $|p, \lambda = 0\rangle$ states which remain singlets.)

However, in theories without the parity symmetry — or even approximate parity symmetry — the particle multiplets with helicities $\pm\lambda$ do not have to pair up. Instead, we may have massless particle species that have only one helicity λ without the opposite helicity $-\lambda$. In particular, in the early version of the Standard Model which had exactly massless neutrinos and antineutrinos, the neutrinos had a single helicity $\lambda = -\frac{1}{2}$ only, while the antineutrinos had the opposite helicity $\lambda = +\frac{1}{2}$ only.

Once the neutrino oscillations were discovered in 2001, the Standard model was modified to give the neutrinos a small mass. (Or rather, a small mass matrix which is non-diagonal in the $(\nu_e, \nu_\mu, \nu_\tau)$ basis, hence the neutrino oscillations.) Consequently, both neutrinos and antineutrinos are massive spin $j = \frac{1}{2}$ particles which have both $\lambda = +\frac{1}{2}$ and $\lambda = -\frac{1}{2}$ helicities. However, all the experimentally observed neutrinos and antineutrinos are ultra-relativistic with $\gamma > 10^6$, and in this regime the weak interactions — which are the only interactions of the neutrinos — involve only the left-handed neutrinos with $\lambda = -\frac{1}{2}$ or the right-handed antineutrinos with $\lambda = +\frac{1}{2}$. The opposite helicity states — the neutrinos with $\lambda = +\frac{1}{2}$ and the antineutrinos with $\lambda = -\frac{1}{2}$ — might exist, but we shall never see them experimentally. Alternatively, we may have Majorana neutrinos (which I shall explain later in class) in which the neutrino and the antineutrino belong to the same spin $j = \frac{1}{2}$ multiplet $(\nu, \bar{\nu})$. At the ultra-relativistic velocities, the $\lambda = -\frac{1}{2}$ state of this spin doublet appear to be a neutrino while the $\lambda = +\frac{1}{2}$ state appears to be an antineutrino.

Tachyons

A *tachyon* is a hypothetical particle moving faster them light. No experiment have ever seen a tachyon, and classically they cannot exist since a particle with initial velocity slower then light cannot be accelerated to a superluminal velocity. However, in quantum mechanics a collision of other particles may create a tachyon whose initial velocity is already faster then light. Then, regardless of the forces acting on such a tachyon its velocity would always stay faster then light, until it is eventually destroyed in some other particle collision.

A tachyon has a spacelike momentum p^μ with $p^0 < 0$, which corresponds to a negative m^2. Indeed, for any particle $\vec{p} = \mathbf{v}p^0$ (in $c = 1$ units), hence for $|\mathbf{v}| > 1$ $|\mathbf{p}| > |p^0$ and consequently $p^2 = (p^0)^2 - \mathbf{p}^2 < 0$. Therefore, a tachyon has no rest frame, but instead there is a frame where the velocity is infinite and the energy vanishes, $p^\mu = (0, \mathbf{p})$.

For example, consider a frame where $p^\mu = (0, 0, 0, p^z)$. The little group of this momentum is generated by the $\hat{J}^{\mu\nu}$ with $\mu, \nu \neq z$, namely $\hat{J}^{xy} = \hat{J}^z$, $\hat{J}^{0x} = \hat{K}^x$, and $\hat{J}^{0y} = \hat{K}^y$, and the little group $G(p)$ itself is $SO^+(2, 1)$ — the Lorentz group in 2 space dimensions (x and y but not z) and one time dimension. Likewise, for any lightlike p^μ the little group $G(p)$ is isomorphic to the ${}^+SO(2, 1)$.

Unlike the simple and compact $SO(3)$ group or the non-simple and non-compact ISO(2) group, the $SO^+(2, 1)$ group is simple but non-compact. Consequently, it does not have any non-trivial finite unitary representations. Physically, this means that the tachyons — if they exist at all — do not have any non-trivial spin or helicity states; instead, they are spinless particles ($j = 0$) with $\lambda = 0$ only. In other words, a particle representation of the Lorentz group for a tachyon species comprises states $|p\rangle$ where p spans the tachyonic mass shell (fixed negative p^2) but there are no spin or polarization quantum numbers of any kind.

Once we go beyond the ordinary quantum mechanics to the quantum field theory, we find that in theories with a stable vacuum state $|\text{vac}\rangle$ the tachyons do not exist. Instead, if we happen to have a field with a seemingly negative m^2, this field gives rise to a vacuum instability rather than to tachyonic quanta. But this issue is outside the scope of these notes Lorentz symmetries, so I shall explain it from the blackboard, or rather from the document camera.

Wigner Theorem in Other Dimensions

The Wigner theorem can be easily generalized to spacetimes of $d > 4$ dimensions and the corresponding $SO^+(d-1,1)$ Lorentz symmetries. Such generalizations are particularly important for the string theory, especially for $d = 26$ for the bosonic string and $d = 10$ for the superstring.

- ⋆ The massive particles with $p^2 = m^2 > 0$ form "spin" multiplets of the $SO(d-1)$ rotation symmetry, or rather the space rotation symmetry boosted to the particle's rest frame, which is isomorphic to the $SO(d-1)$.
- ⋆ The massless particles with $p^2 = 0$ form "helicity" multiplets of the $SO(d-2)$ symmetry of rotations in planes transverse to the particle's velocity. In addition, each state in the multiplet is annihilated by the $\hat{I}^\nu = (p_\mu/E)\hat{J}^{\mu\nu}$ generators analogous to the $\hat{\mathbf{I}}^\perp$ for $d = 4$.
- ∗ Note that for $d = 4$ the $SO(2)$ symmetry is abelian, so all the helicity multiplets are singlets $|\lambda\rangle$, although they are non-trivial singlets for $\lambda \neq 0$. However, in higher dimensions $d > 4$, the $SO(d-2)$ group of transverse rotations becomes non-abelian and develops larger multiplets. For example, for $d = 5$, the $SO(3)$ "helicity" multiplets comprise the $|j, m\rangle$ states with a fixed j and m running from $-j$ to $+j$.
- ∗ Nevertheless, the $SO(d-2)$ "helicity" multiplets of massless particles are smaller than the $SO(d-1)$ "spin" multiplets of the massive particles. For example, in $d = 10$, the massive vector multiplet of the $SO(9)$ has 9 polarization states — 8 transverse and 1 longitudinal, — while the massless vector multiplet of the $SO(8)$ has only the 8 transverse polarizations, by the longitudinal polarization does not exist.
- • In any dimension, if a tachyon exist at all, it must be a scalar without any polarization or spin-like quantum numbers.

For completeness sake, let me also say a few words about the lower spacetime dimensions, namely $d = 2$ and $d = 3$.

- ⋆ In $d = 2$ dimensions (one space one time) there are no meaningful spin or helicity quantum numbers and the particle multiplet of the $SO^+(1,1)$ Lorentz symmetry comprises just the $|p\rangle$ spinless states. Hence, all kinds of fields — scalars, vectors, spinors, *etc.*, — have the same kinds of particles as their quanta.

∗ The massless particles in $d = 2$ have only two values for their velocities, $v = +1$ (move to the right at the speed of light) or $v = -1$ (move to the left at the speed of light), and no continuous boost can change the sign of this velocity. Consequently, there are two kinds of massless particles, the right movers with $p^\mu = (+E, +E)$ and the left movers with $p^\mu = (+E, -E)$, and the continuous Lorentz symmetry do not mix them up.

⋆ In $d = 3$ dimension (two space one time), there is no meaningful $SO(1)$ "helicities", hence the massless photon has only one transverse polarizations and likewise for the other kinds of massless particles.

∗ On the other hand, the massive particles in $d = 3$ do have meaningful $SO(2)$ "spins". However, since the $SO(2)$ group is abelian, all the "spin" multiplets are singlets of definite m (the eigenstate of the $\tilde{J}^{xy}$). Although if we include the parity (a mirror reflection in space) with the Lorentz symmetry, then the states $|{+m}\rangle$ and $|{-m}\rangle$ pair up into doublets.

∗ Also, for the abelian $SO(2)$ "spin" group, the value of m does not have to be integer or half integer but could be any real number. In perturbative field theories, the quanta of fields belonging to specific $SO(2,1)$ Lorentz multiplets always have integer or half0integer m, but in the non-perturbative theories one may get composite particles with fractional "spins" m. Such fractional-spin particles are called *anyons*, and while they are quite rare in relativistic field theories, they are more common in condensed matter. For example, they play important rule on the fractional quantum Hall effect.

Lorentz Multiplets of Fields

Unlike the particle states which form infinite but unitary multiplets of the Lorentz symmetry, the field components form finite but generally non-unitary multiplets, for example the vector multiplet $A^\mu(x)$ or the antisymmetric tensor multiplet $F^{\mu\nu}(x)$. However, when considering the Lorentz transformations of the fields' components into each other, one should also keep in mind that the points x on which the fields depend also transform into each other. For example, under an active Lorentz transform

$$x^\mu \; to \; x'^\mu \;=\; L^\mu{}_\nu x^\nu, \tag{26}$$

the scalar field $\Phi(x)$ transforms into itself but at a new x, namely

$$\Phi'(x') \;=\; \Phi(x). \tag{27}$$

For the vector field — or rather the vector multiplet of the 4 field components $A^\mu(x)$, the transformation involves both changing the position x and transforming the components into each other,

$$A'^\mu(x') \;=\; L^\mu_{\ \nu} A^\nu(x). \tag{28}$$

Likewise, the antisymmetric tensor multiplet of fields $F^{\mu\nu}(x) = -F^{\nu\mu}(x)$ transforms to

$$F'^{\mu\nu}(x') \;=\; L^\mu_{\ \kappa} L^\nu_{\ \lambda} F^{\kappa\lambda}(x), \tag{29}$$

etc., *etc.* Most generally, a Lorentz multiplet of fields have some finite number n of components $\varphi^\alpha(x)$ where α can be any kind of vector, tensor, spinor, *etc.*, index or multi-index. Under a Lorentz symmetry L, the components move to the new location $x' = Lx$ as well as mix with each other according to

$$\varphi'^\alpha(x') \;=\; \sum_\beta M^\alpha_\beta(L) \times \varphi^\beta(x) \tag{30}$$

where $L \to \|M^\alpha_\beta(L)\|$ is some representation of the Lorentz group by $n \times n$ matrices. So our next task is to classify the finite representations of the Lorentz group.

The continuous $SO^+(3,1)$ Lorentz group is related to the 4D rotation group $SO(4)$ by analytic continuation of the real rotations to complex angles. Consequently, any finite representation of the $SO(4)$ group can be analytically continued to an equally finite representation of the Lorentz group, although such analytic continuation will not be unitary any more. Thus, given the list of all the finite $SO(4)$ representations we would immediately get a complete list of finite Lorentz representations. But since many students are probably unfamiliar with the $SO(4)$ representations, let me use a different approach.

Let's reorganize the Lorentz generators $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ into two non-hermitian 3–vectors

$$\hat{\mathbf{J}}_+ \;=\; \tfrac{1}{2}\big(\hat{\mathbf{J}} + i\hat{\mathbf{K}}\big) \quad \text{and} \quad \hat{\mathbf{J}}_- \;=\; \tfrac{1}{2}\big(\hat{\mathbf{J}} - i\hat{\mathbf{K}}\big) \;=\; \hat{\mathbf{J}}_+^\dagger. \tag{31}$$

As you shall see in homework#6, the components of these two vectors obey the $SO(3)\times SO(3)$ like commutation relations:

$$\big[\hat{J}_+^i, \hat{J}_+^j\big] \;=\; i\epsilon^{ijk}\hat{J}_+^k, \quad \big[\hat{J}_-^i, \hat{J}_-^j\big] \;=\; i\epsilon^{ijk}\hat{J}_-^k, \quad \text{but} \quad \big[\hat{J}_+^i, \hat{J}_-^j\big] \;=\; 0. \tag{32}$$

By themselves, the 3 $\hat{J}_+^k$ generate a symmetry group similar to rotations of a 3D space, but since the $\hat{J}_+^k$ are non-hermitian, the finite irreducible multiplets of this symmetry are non-unitary analytic continuations (to complex "angles") of the ordinary angular momentum multiplets (j) of spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. Likewise, the finite irreducible multiplets of the symmetry group generated by the $\hat{J}_-^k$ are analytic continuations of the spin-j multiplets of angular momentum. Moreover, the two symmetry groups commute with each other, so the finite irreducible multiplets of the net Lorentz symmetry are tensor products $(j_+)\otimes(j_-)$ of the $\hat{\mathbf{J}}_+$ and $\hat{\mathbf{J}}_-$ multiplets. In other words, distinct finite irreducible multiplets of the Lorentz symmetry may be labeled by *two* integer or half-integer 'spins' j_+ and j_-, while the states within such a multiplet are $|j_+, j_-, m_+, m_-\rangle$ for $m_+ = -j_+, \ldots, +j_+$ and $m_- = -j_-, \ldots, +j_-$.

For the $SO(3)$ group, the integer-j representations are single-valued by the half-integer-j representations are double-valued, so to make all the representations single-valued we promote the $SO(3)$ group to its double cover $\mathrm{Spin}(3) \cong SU(2)$. In the $\mathrm{Spin}(3)$ group, rotations through angle 4π are trivial but the rotations through angle 2π are not, and they are represented by -1 matrices in the half-integer-j representations. For the Lorentz group we have a similar situation: the representations with integer $j_+ + j_-$ are single-valued while the representations with half-integer $j_+ + j_-$ are double-valued, and to make all of the single valued we promote the $SO^+(3,1)$ group to its double cover $\mathrm{Spin}(3,1)$, which happens to be isomorphic to $SL(2,\mathbf{C})$, the group of complex 2×2 matrices with unit determinants (but without any unitarity requirements).

The simplest non-trivial Lorentz — or rather $\mathrm{Spin}(3,1)$ — representations are two inequivalent doublets, the left-handed Weyl spinor $\mathbf{2}$ and the right-handed Weyl spinor $\overline{\mathbf{2}}$. (The

Dirac spinor we shall study later this week is a reducible representation $\mathbf{4}_{\text{Dirac}} = \mathbf{2}+\overline{\mathbf{2}}$.) The $\mathbf{2}$ representation has $j_+ = \frac{1}{2}$ and $j_- = 0$, so $\hat{\mathbf{J}}_+$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ while $\hat{\mathbf{J}}_-$ does not act at all. Likewise, the $\overline{\mathbf{2}}$ representations has $j_+ = 0$ and $j_- = \frac{1}{2}$, so this time the $\hat{\mathbf{J}}_-$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ while the $\hat{\mathbf{J}}_+$ does not act at all. In terms of the $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ Lorentz generators, in the $\mathbf{2}$ representation $\hat{\mathbf{J}}$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ and $\hat{\mathbf{K}}$ as $-\frac{i}{2}\boldsymbol{\sigma}$, while in the $\overline{\mathbf{2}}$ representation $\hat{\mathbf{J}}$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ and $\hat{\mathbf{K}}$ as $+\frac{i}{2}\boldsymbol{\sigma}$. Consequently a 3D rotation $R(\phi,\mathbf{n})$ through angle ϕ around axis $\mathbf{n}$ is represented in both representations as

$$M_{\mathbf{2}}(R) \;=\; M_{\overline{\mathbf{2}}}(R) \;=\; \exp\left(-\frac{i\phi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right), \tag{33}$$

while a Lorentz boost B of speed v in the direction $\mathbf{n}$ is represented by

$$M_{\mathbf{2}}(B) \;=\; \exp\left(-\frac{r}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right), \quad M_{\overline{\mathbf{2}}}(B) \;=\; \exp\left(+\frac{r}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right) \tag{34}$$

where $r = \operatorname{artanh}(v)$ is the *rapidity* of the boost. For successive boosts in the same direction, the rapidities add up, $r_{1+2} = r_1 + r_2$. Consequently, a finite Lorentz boost of rapidity r in the direction $\mathbf{n}$ is $B = \exp(r\mathbf{n}\cdot\hat{\mathbf{K}})$, hence eqs. (34). The more familiar β and γ parameters of a Lorentz boost are related to the rapidity as

$$\beta \;=\; \tanh(r), \quad \gamma \;=\; \cosh(r), \quad \beta\gamma \;=\; \sinh(r). \tag{35}$$

Note: all the 2×2 matrices (33) and (34) have unit determinants. And since any continuous Lorentz symmetry is a product of a boost and a space rotation, it follows that

$$\forall L \in SO^+(3,1): \quad \det\bigl(M_{\mathbf{2}}(L)\bigr) \;=\; 1 \quad \text{and} \quad \det\bigl(M_{\overline{\mathbf{2}}}(L)\bigr) \;=\; 1. \tag{36}$$

In $SL(2,\mathbf{C}) = \mathrm{Spin}(3,1)$ terms, we may identify the $\mathbf{2}$ representation with the fundamental doublet of the $SL(2,\mathbf{C})$ which transforms as $\xi'^\alpha = M^\alpha_{\ \beta}\xi^\beta$ for $M = M_{\mathbf{2}}(L) \in SL(2,\mathbf{C})$. Consequently, the $\overline{\mathbf{2}}$ representation becomes inequivalent to $\mathbf{2}$ but equivalent to its complex conjugate $\mathbf{2}^*$ comprised of $\eta'_{\dot\alpha} = \overline{M}_{\dot\alpha}^{\ \dot\beta}\eta_{\dot\beta}$ for

$$\overline{M} \;=\; M_{\overline{\mathbf{2}}}(L) \;=\; \sigma_2 M^* \sigma_2 \quad \text{for any } L \text{ and corresponding } M = M_{\mathbf{2}}(L). \tag{37}$$

Or if we raise the dotted $\overline{\mathbf{2}}$ index of the η's using the σ_2 matrix, $\eta^{\dot\alpha} = \sigma_2^{\dot\alpha\dot\gamma}\eta_{\dot\gamma}$, then $\eta'^{\dot\alpha} = M^{*\dot\alpha}_{\ \ \dot\beta}\eta^{\dot\beta}$.

Next, consider the $(j_+ = j_- = \frac{1}{2})$ representation, which happens to be equivalent to the Lorentz vector representation, $V^\mu \to V'^\mu = L^\mu_\nu V^\nu$. In the $SL(2,\mathbf{C})$ terms, this $(j_+ = j_- = \frac{1}{2})$ representation is bi-spinor $V^{\alpha,\dot\beta}$ which transforms as a product of $\mathbf{2}$ and $\overline{\mathbf{2}}$ spinors,

$$V'^{\alpha\dot\gamma} \;=\; M^\alpha_\beta {M^*}^{\dot\gamma}_{\dot\delta} V^{\beta\dot\delta}, \tag{38}$$

or in the matrix language

$$V' \;=\; M \times V \times M^\dagger. \tag{39}$$

The map between bi-spinors and Lorentz vectors involves four hermitian 2×2 matrices $\sigma_\mu = (1, \boldsymbol{\sigma})$. Using the σ_μ, we may re-cast any Lorentz vector V^μ as a matrix

$$V^\mu \;\to\; V^\mu \sigma_\mu \;=\; V^0 \;+\; \mathbf{V}\cdot\boldsymbol{\sigma} \tag{40}$$

an hence as a $\left(\frac{1}{2}, \frac{1}{2}\right)$ bi-spinor

$$V^{\alpha\dot\gamma} \;=\; \left(V^\mu \sigma_\mu\right)^{\alpha\dot\gamma} \;=\; V^0 \delta_{\alpha\dot\gamma} \;+\; \mathbf{V}\cdot\boldsymbol{\sigma}_{\alpha\dot\gamma}\,. \tag{41}$$

Eq. (39) defines a linear transform of 2×2 matrices $V \to V'$, and since the four matrices σ_μ form a complete basis of such 2×2 matrices, eq. (39) also defines a linear transform of the corresponding Lorentz vectors, $V'^\mu = L^\mu_\nu(M) V^\nu$. In your homework#6 you shall prove that this transform is is real (real V'^μ for real V^μ), Lorentzian (preserves $V'_\mu V'^\mu = V_\mu V^\mu$), orthochronous, and proper ($\det(L) = +1$). And that what makes the the bi-spinor $(j_+ = j_- = \frac{1}{2})$ representation of $SL(2,\mathbf{C})$ equivalent to the vector representation of the Lorentz group. Also, eqs. (39) and (40) provide us with a reverse map from the $SL(2,\mathbf{C})$ group acting on the Weyl spinors $\mathbf{2}$ and $\overline{\mathbf{2}}$ back to the Lorentz group acting on vectors, tensors, *etc.*.

In general, any (j_+, j_-) multiplet of the $SL(2,\mathbf{C})$ with integer net spin $j_+ + j_-$ is equivalent to some kind of a Lorentz tensor. (Here, we include the scalar and the vector among the tensors.) For example, the $(1,1)$ multiplet is equivalent to a symmetric, traceless 2–index tensor $T^{\mu\nu} = +T^{\nu\mu}$, $T^\mu_\mu = 0$. But to save time, I leave the proof of this equivalence to your homework#6.

For $j_+ \neq j_-$ the (j_+, j_-) representation of $SL(2, \mathbf{C})$ is complex, so relating it to a real Lorentz tensor is tricky. The trick here is to use two $SL(2, \mathbf{C})$ multiplets with opposite j_+ and j_-, which makes them complex conjugates of each other. Consequently, combining this two conjugate multiplets together makes for a real multiplet, and it is this real multiplet which is equivalent to a Lorentz Tensor. For example, the $(j_+ = 1, j_- = 0)$ and the $(j_+ = 0, j_- = 1)$ representations are complex conjugates of each other, and together the 6-member real representation $(1, 0) + (0, 1)$ is equivalent to the antisymmetric 2–index Lorentz tensor $F^{\mu\nu} = -F^{\nu\mu}$. Again, proving this equivalence is a part of homework#6.