The Quantum EM Fields and the Photon Propagator

Quantizing the *free* electromagnetic tension fields $\mathbf{E}$ and $\mathbf{B}$ is fairly straightforward. The time-independent Maxwell equations

$$\nabla\cdot\hat{\mathbf{B}}(\mathbf{x}) \;=\; 0, \qquad \nabla\cdot\hat{\mathbf{E}}(\mathbf{x}) \;=\; \hat{J}^0(\mathbf{x}) \;\to\; 0 \quad \text{(for the free fields)} \tag{1}$$

are imposed as operatorial constraints in the Hilbert space, while the time-dependent Maxwell equations

$$\frac{\partial}{\partial t}\hat{\mathbf{B}} \;=\; -\nabla\times\hat{\mathbf{E}}, \qquad \frac{\partial}{\partial t}\hat{\mathbf{E}} \;=\; +\nabla\times\hat{\mathbf{B}} \tag{2}$$

follow from the free Hamiltonian

$$\hat{H} \;=\; \int\! d^3\mathbf{x}\Big(\tfrac{1}{2}\hat{\mathbf{E}}^2(\mathbf{x}) \,+\, \tfrac{1}{2}\hat{\mathbf{B}}^2(\mathbf{x})\Big) \tag{3}$$

and the equal-time commutation relations

$$\begin{aligned}
\big[\hat{E}^i(\mathbf{x},t),\hat{E}^j(\mathbf{y},t)\big] \;&=\; 0,\\
\big[\hat{B}^i(\mathbf{x},t),\hat{B}^j(\mathbf{y},t)\big] \;&=\; 0,\\
\big[\hat{E}^i(\mathbf{x},t),\hat{B}^j(\mathbf{y},t)\big] \;&=\; -i\epsilon^{ijk}\frac{\partial}{\partial x^k}\delta^{(3)}(\mathbf{x}-\mathbf{y}).
\end{aligned} \tag{4}$$

In light of eqs. (1), we may expand the vector fields into momentum/polarization modes using *only the transverse polarizations*, for example the helicity modes $\lambda=\pm1$ only, thus

$$\begin{aligned}
\mathbf{k}\times\mathbf{e}_{\mathbf{k},\lambda} \;&=\; i\lambda\,|\mathbf{k}|\;\mathbf{e}_{\mathbf{k},\lambda}, \qquad \mathbf{e}_{\mathbf{k},\lambda}\perp\mathbf{k} \quad \text{for } \lambda=\pm1,\\
\hat{\mathbf{E}}(\mathbf{x}) \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{\lambda=\pm1\,\text{only}} e^{i\mathbf{x}\mathbf{k}}\mathbf{e}_{\mathbf{k},\lambda}\times\hat{E}_{\mathbf{k},\lambda}\,, \qquad \hat{\mathbf{E}}^\dagger_{\mathbf{k},\lambda} \;=\; -\hat{\mathbf{E}}_{-\mathbf{k},\lambda}\,,\\
\hat{\mathbf{B}}(\mathbf{x}) \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{\lambda=\pm1\,\text{only}} e^{i\mathbf{x}\mathbf{k}}\mathbf{e}_{\mathbf{k},\lambda}\times\hat{B}_{\mathbf{k},\lambda}\,, \qquad \hat{\mathbf{B}}^\dagger_{\mathbf{k},\lambda} \;=\; -\hat{\mathbf{B}}_{-\mathbf{k},\lambda}\,,\\
\big[\hat{\mathbf{E}}_{\mathbf{k},\lambda},\hat{\mathbf{E}}_{\mathbf{k}',\lambda'}\big] \;=\; 0, \quad &\big[\hat{\mathbf{B}}_{\mathbf{k},\lambda},\hat{\mathbf{B}}_{\mathbf{k}',\lambda'}\big] \;=\; 0, \quad \big[\hat{\mathbf{E}}_{\mathbf{k},\lambda},\hat{\mathbf{B}}_{\mathbf{k}',\lambda'}\big] \;=\; i\lambda|\mathbf{k}|\delta_{\lambda\lambda'}\times(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}').
\end{aligned} \tag{5}$$

Consequently, the photonic creation and annihilation operators are defined for the transverse polarizations only,

$$\hat{a}_{\mathbf{k},\lambda} \;=\; \lambda\hat{B}_{\mathbf{k},\lambda} \,+\, i\hat{E}_{\mathbf{k},\lambda}\,, \quad \hat{a}^\dagger_{\mathbf{k},\lambda} \;=\; \lambda\hat{B}^\dagger_{\mathbf{k},\lambda} \,-\, i\hat{E}^\dagger_{\mathbf{k},\lambda}\,, \quad \lambda \;=\; \pm1 \text{ only}, \tag{6}$$

$$\left[\hat{a}_{\mathbf{k},\lambda},\hat{a}_{\mathbf{k}',\lambda'}\right] \;=\; 0,\quad \left[\hat{a}^\dagger_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}\right] \;=\; 0,\quad \left[\hat{a}_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}\right] \;=\; \delta_{\lambda\lambda'}\times 2|\mathbf{k}|(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{7}$$

In terms of these operators, the free EM Hamiltonian becomes

$$\hat{H}\;\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\sum_{\lambda=\pm1\,\mathrm{only}}\omega_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}\;+\;\mathrm{const}\quad\mathrm{for}\quad \omega_{\mathbf{k}}\;=\;+|\mathbf{k}|\,, \tag{8}$$

which makes for the usual time-dependence in the Heisenberg picture:

$$\hat{a}_{\mathbf{k},\lambda}\;\to\;\hat{a}_{\mathbf{k},\lambda}\times e^{-i\omega_{\mathbf{k}}t},\quad \hat{a}^\dagger_{\mathbf{k},\lambda}\;\to\;\hat{a}^\dagger_{\mathbf{k},\lambda}\times e^{+i\omega_{\mathbf{k}}t}, \tag{9}$$

and hence (after a bit of algebra)

$$\begin{aligned}\hat{\mathbf{E}}(\mathbf{x},t) &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega}\sum_{\lambda=\pm1\,\mathrm{only}}\Bigl(e^{-i\omega t+i\mathbf{kx}}(-i\omega\mathbf{e}_{\mathbf{k},\lambda})\hat{a}_{\mathbf{k},\lambda}\;+\;e^{+i\omega t-i\mathbf{kx}}(+i\omega\mathbf{e}^*_{\mathbf{k},\lambda})\hat{a}^\dagger_{\mathbf{k},\lambda}\Bigr)\,,\\ \hat{\mathbf{B}}(\mathbf{x},t) &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega}\sum_{\lambda=\pm1\,\mathrm{only}}\Bigl(e^{-i\omega t+i\mathbf{kx}}(-i\mathbf{k}\times\mathbf{e}_{\mathbf{k},\lambda})\hat{a}_{\mathbf{k},\lambda}\;+\;e^{+i\omega t-i\mathbf{kx}}(+i\mathbf{k}\times\mathbf{e}^*_{\mathbf{k},\lambda})\hat{a}^\dagger_{\mathbf{k},\lambda}\Bigr)\,.\end{aligned} \tag{10}$$

Up to now I have focused on the EM tension fields $\hat{F}^{\mu\nu}(x)$, but to couple the EM to come charged fields — like the electron field $\hat{\Psi}(x)$ — I would also need the quantum potential fields $\hat{A}_\mu(x)$ to spell out the interactions $\mathcal{L}\supset -J_\mu\times A^\mu$. Classically, the potential fields are subject to gauge symmetries

$$A^\mu_{\mathrm{old}}(x)\;\to\;A^\mu_{\mathrm{new}}(x)\;=\;A^\mu_{\mathrm{old}}(x)\;-\;\partial^\mu\Lambda(x), \tag{11}$$

but implementing such symmetries in the quantum theory is highly non-trivial since a transform changing the time-dependence of quantum fields must change the Hamiltonian operator. There are more problems with quantum gauge theories, and I'll deal with them next semester.

For the moment, let me simply say that the *canonical* quantization of the potential fields $A^\mu(x)$ requires *fixing a gauge.* That is, we should remove the gauge-redundancy of the potential fields by imposing a local linear constraint at each x, for example $\nabla\cdot\mathbf{A}(x)\equiv 0$ (the Coulomb gauge), or $\partial_\mu A^\mu(x)\equiv 0$ (the Landau gauge), or $A^3(x)\equiv 0$ (the axial gauge).

Once we impose such a constraint at the Lagrangian level, we find the canonical conjugates of the remaining independent fields, build the classical Hamiltonian, and then quantize the theory in the usual way to build the quantum fields, the Hilbert space where they act, and the Hamiltonian operator. In general, different gauge-fixing constraints give rise to different quantum theories, each having its own Hilbert space and the Hamiltonian operator. However, all such theories are physically equivalent to each other!

For the *free* electromagnetic fields, I do not have to re-quantize the theory starting from the Lagrangian and some gauge-fixing constraint for the $A^\mu(x)$. Since I already have the quantum $\hat{\mathbf{E}}(x)$ and $\hat{\mathbf{B}}(x)$ fields as in eqs. (10), I can can obtain the free potential fields $\hat{A}^\mu(x)$ by simply solving the equations $\partial^\mu \hat{A}^\nu(x) - \partial^\nu \hat{A}^\mu(x) = \hat{F}^{\mu\nu}(x)$ *combined with the gauge-fixing constraint.* By linearity and translation invariance, the result has form

$$\hat{A}^\mu_{\text{free}}(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega}\sum_{\lambda=\pm1\,\text{only}}\Big(e^{-ikx}\mathcal{E}^\mu(\mathbf{k},\lambda)\times\hat{a}_{\mathbf{k},\lambda}\;+\;e^{+ikx}\mathcal{E}^{\mu*}(\mathbf{k},\lambda)\times\hat{a}^\dagger_{\mathbf{k},\lambda}\Big) \tag{12}$$

where the plane waves

$$A^\mu(x) \;=\; e^{-ikx}\times\mathcal{E}^\mu(\mathbf{k},\lambda) \quad\text{and}\quad A^\mu(x) \;=\; e^{+ikx}\times\mathcal{E}^{\mu*}(\mathbf{k},\lambda) \quad\text{for}\quad k^0 = +|\mathbf{k}| \tag{13}$$

obey the gauge-fixing conditions as well as the Maxwell equations. For example, in the Coulomb gauge $\nabla\cdot\hat{\mathbf{A}} = 0$, the polarization vectors $\mathcal{E}^\mu$ have simple form

$$\mathcal{E}^\mu(\mathbf{k},\lambda) \;=\; \big(0,\mathbf{e}(\mathbf{k},\lambda)\big) \tag{14}$$

while in the axial gauge $\mathbf{n}\cdot\mathbf{A} = 0$ (where $\mathbf{n}$ is a fixed unit 3-vector) we have much messier formulae

$$\vec{\mathcal{E}}(\mathbf{k},\lambda) \;=\; \mathbf{e}(\mathbf{k},\lambda) \;-\; \frac{\mathbf{n}\cdot\mathbf{e}(\mathbf{k},\lambda)}{\mathbf{n}\cdot\mathbf{k}}\,\mathbf{k},\quad \mathcal{E}^0(\mathbf{k},\lambda) \;=\; -\frac{\mathbf{n}\cdot\mathbf{e}(\mathbf{k},\lambda)}{\mathbf{n}\cdot\mathbf{k}}\,\omega_{\mathbf{k}}\,. \tag{15}$$

PHOTON PROPAGATOR

The photon propagator

$$G_F^{\mu\nu}(x-y) \;=\; \langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \tag{16}$$

depends on the gauge-fixing condition for the quantum potential fields $\hat{A}^\mu(x)$. So let me first calculate it for the Coulomb gauge $\nabla\cdot\hat{\mathbf{A}}\equiv 0$, and then I'll deal with the other gauges.

Instead of calculating the propagator directly from eqs. (16), (12), and (14), let me use the fact that $G^{\mu\nu}(x-y)$ is a Green's function of the Maxwell equations for the A^μ fields,

$$\partial^2 A^\nu(x) \;-\; \partial^\nu(\partial_\mu A^\mu(x)) \;=\; J^\nu(x) \quad\Longrightarrow\quad A^\mu(x) \;=\int\! d^4y\,(-i)G^{\mu\nu}(x-y)\times J_\nu(y). \tag{17}$$

Let's start with the scalar potential $A^0(\mathbf{x},t)$. In the Coulomb gauge we have

$$\partial_\mu A^\mu \;=\; \partial_0 A^0 \;+\; \nabla\cdot\mathbf{A} \;=\; \partial_0 A^0 \;+\; 0, \tag{18}$$

which makes the equation for the $A^0(x)$ time-independent. Indeed,

$$J^0 \;=\; \partial^2 A^0(x) \;-\; \partial^0(\partial_\mu A^\mu(x)) \;=\; (\partial_0^2-\nabla^2)A^0 \;-\; \partial^0(\partial_0 A^0) \;=\; -\nabla^2 A^0(x) \tag{19}$$

while the terms containing the time derivatives ∂_0 cancel out. Consequently, the $A^0(\mathbf{x},t)$ is the *instantaneous* Coulomb potential for the electric charge density $J^0(\mathbf{y},t)$,

$$A^0(\mathbf{x},t) \;=\int\! d^3\mathbf{y}\,\frac{J^0(\mathbf{y},\,\text{same}\,t)}{4\pi|\mathbf{x}-\mathbf{y}|} \tag{20}$$

— which is why this gauge is called the Coulomb gauge. In terms of the Green's function, this means

$$G^{0i} \;\equiv\; 0, \quad G^{00}(x-y) \;=\; \frac{i\delta(x^0-y^0)}{4\pi|\mathbf{x}-\mathbf{y}|}\,, \tag{21}$$

of after Fourier transform to the momentum space,

$$G^{00}(k) \;=\; \frac{i}{\mathbf{k}^2} \quad\text{independent of } k^0. \tag{22}$$

As to the vector potential $\mathbf{A}$, it obeys

$$\partial^2 \mathbf{A} \;+\; \nabla(\partial_\mu A^\mu) \;=\; \mathbf{J} \tag{23}$$

where in the Coulomb gauge

$$\partial_\mu A^\mu \;=\; \partial_0 A^0 \;=\; -\frac{1}{\nabla^2}\,\partial_0 J^0 \;=\; +\frac{1}{\nabla^2}\,(\nabla\cdot\mathbf{J}) \tag{24}$$

where the last equality follows from the electric current conservation, $\partial_0 J^0 = -\nabla\cdot\mathbf{J}$. Consequently,

$$(\partial_0^2 - \nabla^2)\mathbf{A} \;=\; \mathbf{J} \;-\; \nabla\,\frac{1}{\nabla^2}\,(\nabla\cdot\mathbf{J}), \tag{25}$$

or in momentum basis

$$-(k_0^2 - \mathbf{k}^2)\times A^i(k) \;=\; J^i \;-\; \frac{k^i k^j}{\mathbf{k}^2}\times J^j. \tag{26}$$

In terms of the Green's function, this means

$$G^{i0} \;=\; 0, \quad G^{ij}(k) \;=\; \frac{i}{k_0^2 - \mathbf{k}^2}\times\left(\delta^{ij} \;-\; \frac{k^i k^j}{\mathbf{k}^2}\right). \tag{27}$$

Altogether, all the components of the $G^{\mu\nu}(k)$ can be summarized as

$$G^{\mu\nu}(k) \;=\; \frac{i}{k_0^2 - \mathbf{k}^2}\times C^{\mu\nu}(k) \tag{28}$$

where

$$C^{ij} \;=\; \delta^{ij} \;-\; \frac{k^i k^j}{\mathbf{k}^2}\,, \qquad C^{i0} \;=\; C^{0i} \;=\; 0, \qquad C^{00} \;=\; \frac{k_0^2 - \mathbf{k}^2}{\mathbf{k}^2} \;=\; \frac{k_0^2}{\mathbf{k}^2} \;-\; 1. \tag{29}$$

A convenient way to summarize this $C^{\mu\nu}(k)$ tensor is

$$C^{\mu\nu}(k) \;=\; -g^{\mu\nu} \;+\; k^\mu c^\nu(k) \;+\; k^\nu c^\mu(k) \qquad \text{for} \quad c^\mu(k) \;=\; \frac{(k^0, -\mathbf{k})}{2\mathbf{k}^2}\,. \tag{30}$$

In the coordinate space

$$G^{\mu\nu}(x-y) \;=\; \int\!\frac{d^4k}{(2\pi)^4}\,\frac{ie^{-ik(x-y)}}{k^2}\times C^{\mu\nu}(k) \tag{31}$$

where the denominator has poles at $k^0 = \pm|\mathbf{k}|$. As usual, these poles should be regularized by moving them off the real axis, and different regularizations lead to different Green's functions. We are interested in the specific Green's function — the Feynman propagator — so we should move the poles to $k^0 = +|\mathbf{k}| - i0$ and $k^0 = -|\mathbf{k}| + i0$, thus

$$G_F^{\mu\nu}(x-y) \;=\; \int\!\frac{d^4k}{(2\pi)^4}\,\frac{ie^{-ik(x-y)}}{k^2+i0} \times \Big(C^{\mu\nu}(k) \;=\; -g^{\mu\nu} \,+\, k^\mu c^\nu(k) \,+\, k^\nu c^\mu(k)\Big). \tag{32}$$

Now consider photon propagators for different gauge conditions for the EM potential fields $\hat{A}^\mu(x)$. When we change a gauge condition, the potential fields change as

$$\hat{A}^\mu_{\text{new}}(x) \;=\; \hat{A}^\mu_{\text{old}}(x) \;-\; \partial^\mu\hat{\Lambda}(x) \tag{33}$$

for some $\hat{\Lambda}(x)$ that depends on the old $\hat{A}^\mu(x)$ and the new gauge condition. For linear gauge conditions,

$$\hat{\Lambda}(x) \;=\; \int\!d^4y\, L_\mu(x-y)\hat{A}^\mu_{\text{old}}(y) \tag{34}$$

for some kernel $L_\mu(x-y)$, or in momentum space

$$\hat{\Lambda}(k) \;=\; L_\mu(k)\hat{A}^\mu_{\text{old}}(k). \tag{35}$$

Consequently, in the new gauge the Feynman propagator becomes

$$\begin{aligned}
G^{\mu\nu}_{\text{new}}(x-y) \;&=\; \langle 0|\,\mathbf{T}\hat{A}^\mu_{\text{new}}(x)\hat{A}^\nu_{\text{new}}(y)\,|0\rangle \\
&=\; \langle 0|\,\mathbf{T}\hat{A}^\mu_{\text{old}}(x)\hat{A}^\nu_{\text{old}}(y)\,|0\rangle \;-\; \frac{\partial}{\partial x_\mu}\,\langle 0|\,\mathbf{T}\hat{\Lambda}(x)\hat{A}^\nu_{\text{old}}(y)\,|0\rangle \\
&\qquad -\; \frac{\partial}{\partial y_\nu}\,\langle 0|\,\mathbf{T}\hat{A}^\mu_{\text{old}}(x)\hat{\Lambda}(y)\,|0\rangle \;+\; \frac{\partial^2}{\partial x_\mu \partial y_\nu}\,\langle 0|\,\mathbf{T}\hat{\Lambda}(x)\hat{\Lambda}(y)\,|0\rangle \\
&=\; G^{\mu\nu}_{\text{old}}(x-y) \;+\; \frac{\partial}{\partial x_\mu}f^\nu(x-y) \;+\; \frac{\partial}{\partial y_\nu}f^\mu(y-x) \;+\; \frac{\partial^2}{\partial x_\mu \partial y_\nu}h(x-y)
\end{aligned}$$

for some functions $f^\mu(x-y)$ and $h(x-y)$. Fourier transforming to the momentum space, we obtain

$$G^{\mu\nu}_{\text{new}}(k) \;=\; G^{\mu\nu}_{\text{old}}(k) \;+\; ik^\mu \times f^\nu(k) \;-\; ik^\nu \times f^{\mu *}(k) \;+\; k^\mu k^\nu \times h(k) \tag{36}$$

for some functions $f^\nu(k) =$ and $h(k)$ of the off-shell momenta. (In terms of the $L_\mu(k)$ from eq. (35), $f^\mu(k) = L_\lambda(k) \times G^{\lambda\nu}_{\text{old}}(k)$ while $h(k) \;=\; L_\kappa(k)L_\lambda(k)G^{\kappa\lambda}_{\text{old}}(k)$.) Consequently, **if** in the old gauge we had

$$G^{\mu\nu}_{\rm old}(k) \;=\; \frac{i}{k^2+i0}\times\Bigl(-g^{\mu\nu}\;+\;k^\mu\times c^\nu_{\rm old}(k)\;+\;k^\nu\times c^{\mu*}_{\rm old}(k)\Bigr) \tag{37}$$

then in the new gauge we also have

$$G^{\mu\nu}_{\rm new}(k) \;=\; \frac{i}{k^2+i0}\times\Bigl(-g^{\mu\nu}\;+\;k^\mu\times c^\nu_{\rm new}(k)\;+\;k^\nu\times c^{\mu*}_{\rm new}(k)\Bigr) \tag{38}$$

for

$$c^\nu_{\rm new}(k) \;=\; c^\nu_{\rm old}(k)\;+\;k^2\times\bigl(f^\nu(k)\;+\;\tfrac{1}{2}h(k)k^\nu\bigr). \tag{39}$$

Now, back in eq. (32) we saw that in the Coulomb gauge the photon propagator indeed has form (37) for some $c^\nu(k)$. Consequently, in any gauge, the photon's Feynman propagator has form

$$G^{\mu\nu}_F(x-y) \;=\; \int\!\frac{d^4k}{(2\pi)^4}\,\frac{ie^{-ik(x-y)}}{k^2+i0}\times\Bigl(-g^{\mu\nu}\;+\;k^\mu c^\nu(k)\;+\;k^\nu c^\mu(k)\Bigr) \tag{40}$$

where the vector-valued function $c^\nu(k)$ depends on the particular gauge condition, but everything else is the same in all gauges.

In the Feynman rules for QED — or for any other QFT containing the EM fields — the photon propagators depend on the gauge-fixing condition via the $c^\nu(k)$ functions. However, all the physical scattering amplitudes turn out to be gauge-independent. We shall see a few examples of such gauge-invariant amplitudes this semester, and I shall prove the general theorem in the Spring.

In practice, one uses whatever gauge condition would simplify the calculation at hand. Usually, this means one of the Lorentz-invariant gauges where $c^\nu(k)$ is parallel to the k^ν and hence

$$G^{\mu\nu}_F(x-y) \;=\; \int\!\frac{d^4k}{(2\pi)^4}\,\frac{ie^{-ik(x-y)}}{k^2+i0}\times\left(-g^{\mu\nu}\;+\;(1-\xi)\frac{k^\mu k^\nu}{k^2+i0}\right) \tag{41}$$

for some real constant ξ. For $\xi=0$ this photon propagator corresponds to the Landau gauge $\partial_\mu A^\mu\equiv 0$. The propagators with other values of ξ do not correspond to any single gauge condition for the $A^\mu(x)$; instead, they obtain from a gauge-averaging procedure I shall explain in April; for the moment, let me simply say that these gauges work fine. Of particular importance is the Feynman gauge $\xi = 1$, which makes for a particularly simple photon propagator

$$G_F^{\mu\nu}(x-y) \;=\; \int\frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\mu\nu}}{k^2+i0}\times e^{-ik(x-y)}. \tag{42}$$

For the rest of this semester — and also through most of the Spring semester — I shall be using this gauge.

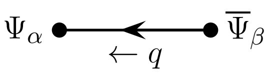
QED Feynman rules

Quantum Electro-Dynamics or QED is the theory of EM field $A_\mu(x)$ coupled to the electron field $\Psi(x)$ (and optionally other charged fermion fields). The Lagrangian is

$$\begin{aligned}\mathcal{L} \;&=\; -\tfrac14 F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{D}-m)\Psi \\ &=\; -\tfrac14 F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{\partial}-m)\Psi + \; eA_\mu\times\overline{\Psi}\gamma^\mu\Psi\end{aligned} \tag{43}$$

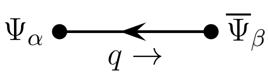
where the first 2 terms on the last line describe free photons and electrons $e^\pm$, and the third term is treated as a perturbation.

The two different field types have different propagators $S^F_{\alpha\beta}(x-y) = \langle 0|\,\mathbf{T}\hat\Psi_\alpha(x)\hat{\overline{\Psi}}_\beta(y)\,|0\rangle$ and $G_F^{\mu\nu}(x-y) = \langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle$. In QED Feynman rules, these propagators are denoted by two different types of internal lines: The electron propagator is drawn as a solid line with an arrow indicating which end of the line belongs to a Ψ field and which to a $\overline{\Psi}$,



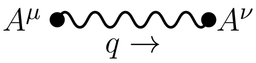
$$= \left[\frac{i}{\not{q}-m+i0}\right]_{\alpha\beta}. \tag{44}$$

The smaller arrow near q indicates the direction of the momentum flow. Both arrows should have the same direction; otherwise we would have



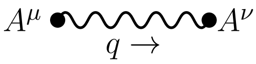
$$= \left[\frac{i}{-\not{q}-m+i0}\right]_{\alpha\beta}. \tag{45}$$

The photon propagator is drawn as a wavy line without arrow,



$$\begin{aligned} A^\mu \sim\sim A^\nu \;&=\; \frac{i}{q^2+i0}\times\Big(-g^{\mu\nu}\,+\,q^\mu c^\nu(q)\,+\,q^\nu c^{\mu*}(q)\Big)\quad\text{for some } c^\nu(q),\\ \text{usually}\;&=\;\frac{-i}{q^2+i0}\times\left(g^{\mu\nu}\,+\,(\xi-1)\frac{q^\mu q^\nu}{q^2+i0}\right)\quad\text{for some constant }\xi. \end{aligned} \tag{46}$$

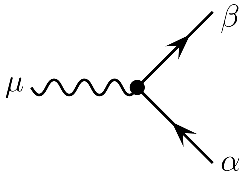
In this class I shall work in the Feynman gauge $\xi = 1$ where the photon propagator is simply



$$A^\mu \sim\sim A^\nu \;=\; \frac{-ig^{\mu\nu}}{q^2+i0}\,. \tag{47}$$

The physical amplitudes do not depend on the gauge-fixing parameters ξ or $c^\nu(q)$, but the intermediate results often do. Make sure to use the same gauge for all the photon propagators in a diagram and also for all the diagrams contributing any particular process.

The vertices of Feynman diagrams follow from the interaction terms in the Lagrangian that involve 3 or more fields. The QED Lagrangian has only one interaction term $eA_\mu \times \overline{\Psi}\gamma^\mu\Psi$, so there is only one vertex type, namely



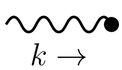
$$=\;(+ie\gamma^\mu)_{\beta\alpha}\,. \tag{48}$$

This vertex has valence = 3, and the 3 lines it connects must be of specific types: one wavy (photonic) line, one solid line with incoming arrow, and one solid line with outgoing arrow.

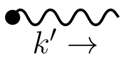
Now consider the external lines. When the quantum EM field

$$\hat{A}^\mu(x)\;=\;\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_{\lambda=\pm1}\Big(e^{-ikx}\,\mathcal{E}^\mu(\mathbf{k},\lambda)\times\hat{a}_{\mathbf{k},\lambda}\,+\,e^{+ikx}\,\mathcal{E}^{\mu*}(\mathbf{k},\lambda)\times\hat{a}^\dagger_{\mathbf{k},\lambda}\Big)^{k^0=+\omega_\mathbf{k}} \tag{49}$$

is contracted with an incoming or an outgoing photon, we end up with an external-line factor accompanying the matching $\hat{a}_{\mathbf{k},\lambda}$ or $\hat{a}^\dagger_{\mathbf{k}',\lambda'}$ operator, namely $e^{-ikx}\times\mathcal{E}^\mu(\mathbf{k},\lambda)$ for an incoming photon or $e^{+ik'x}\times\mathcal{E}^{\mu*}(\mathbf{k}',\lambda')$ for an outgoing photon. The momentum space Feynman rules take care of the $e^{\mp ikx}$ factors, which leaves us with the polarization vectors $\mathcal{E}^\mu(\mathbf{k},\lambda)$ or $\mathcal{E}^{\mu*}(\mathbf{k}',\lambda')$ for the external photon lines:

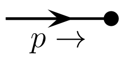


$$=\;\mathcal{E}^\mu(\mathbf{k},\lambda)\qquad\text{for an incoming photon,} \tag{50}$$

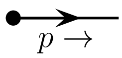


$$= \mathcal{E}^{\mu *}(\mathbf{k}', \lambda') \quad \text{for an outgoing photon.} \tag{51}$$

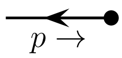
In general, the Feynman rules for any theory has similar external-line factors for all the incoming and outgoing particles. For the scalar particles such factors are trivial — that's why we have not seen them before — but for all other types of particles there are non-trivial external line factors we should not forget. In particular, the fermionic external lines carry plane-wave Dirac spinors $u(p,s)$, $v(p,s)$, $\bar{u}(p,s)$, or $\bar{v}(p,s)$, depending on the charge of the fermion and whether it's incoming or outgoing. Specifically,

an incoming electron e^- carries 

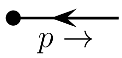
$= u_\alpha(p,s)$, (52)

an outgoing electron e^- carries 

$= \bar{u}_\alpha(p,s)$, (53)

an incoming positron e^+ carries 

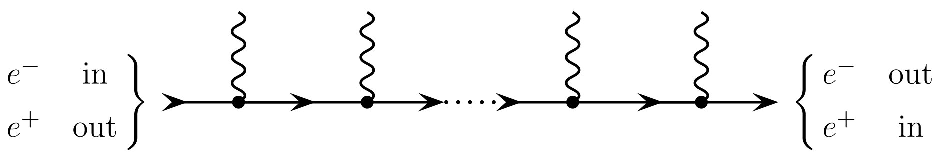
$= \bar{v}_\alpha(p,s)$, (54)

an outgoing positron e^+ carries 

$= v_\alpha(p,s)$. (55)

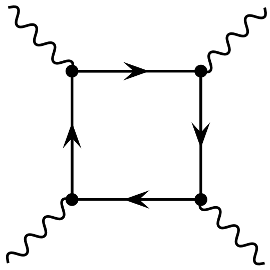
Note that for the incoming / outgoing electrons, the arrow on the external line has the same direction as the particle — incoming for an incoming e^- and outgoing for an outgoing e^-, — but for the positrons the arrow points in the opposite direction from the particle and its momentum: An incoming e^+ has an outgoing line (but in-flowing momentum) while an outgoing e^+ has an incoming line (but out-flowing momentum). In general, the arrows in the fermionic lines follow the flow of the electric charge (in units of $-e$), hence opposite directions for the electrons and the positrons.

Each QED vertex (48) has one incoming fermionic line and one outgoing, and we may think of them as being two segments of single continuous line going through the vertex. From this point of view, a fermionic line enters a diagram as an incoming e^- or an outgoing e^+, goes through a sequence of vertices and propagators, and eventually exits the diagram as an outgoing e^- or an incoming e^+,



(56)

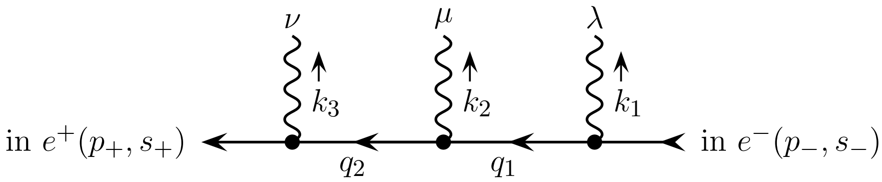
(The photonic lines here may be external or internal; if internal, they connect to some other fermionic lines, or maybe even to the same line at another vertex.) Alternatively, a fermionic line may form a closed loop, for example



(57)

The continuous fermionic lines such as (56) or (57) are convenient for handling the Dirac indices of vertices, fermionic propagators, and external lines. For an open line such as (56), the rule is to read the line in the direction of the arrows, from the line's beginning to its end, spell all the vertices, the propagators, and the external line factors in the same order *right-to-left*, them multiply them together as Dirac matrices. For example, consider a diagram where an incoming electron and incoming positron annihilate into 3 photons, real or virtual. This diagram has a fermionic line which starts at the incoming e^-, goes through 3 vertices and 2 propagators, and exits at the incoming e^+ as shown below:

(58)



The propagators here carry momenta $q_1 = p_- - k_1$ and $q_2 = q_1 - k_2 = k_3 - p_+$. The fermionic line (58) carries the following factors:

- $u(p_-, s_-)$ for the incoming e^-;
- $+ie\gamma^\lambda$ for the first vertex (from the right);
- $\dfrac{i}{\not{q}_1 - m + i0}$ for the first propagator;
- $+ie\gamma^\mu$ for the second vertex;
- $\dfrac{i}{\not{q}_2 - m + i0}$ for the second propagator;
- $+ie\gamma^\nu$ for the third vertex;

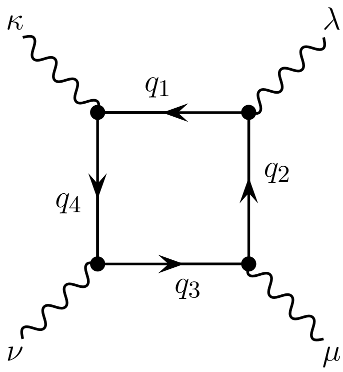
- $\bar{v}(p_+, s_+)$ for the incoming e^+.

Reading all these factors in the order of the line (58), tail-to-head, and multiplying them right-to-left, we get the following Dirac 'sandwich'

$$\bar{v}(p_+, s_+) \times (+e\gamma^\nu) \times \frac{i}{\not{q}_2 - m + i0} \times (+ie\gamma^\mu) \times \frac{i}{\not{q}_1 - m + i0} \times (+ie\gamma^\lambda) \times u(p_-, s_-). \quad (59)$$

In this formula, all the Dirac indices are suppressed; the rule is to multiply all factors as Dirac matrices (or row / column spinors) *in this order.*

For a closed fermionic loop such as (57), the rule is to start at an arbitrary vertex or propagator, follow the line until one gets back to the starting point, multiply all the vertices and the propagators *right-to-left in the order of the line, then take the trace of the matrix product.* For example, the loop



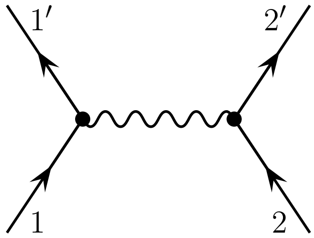
(60)

produces the Dirac trace

$$\mathrm{tr}\Big[(+ie\gamma^\kappa) \times \frac{i}{\not{q}_1 - m + i0} \times (+ie\gamma^\lambda) \times \frac{i}{\not{q}_2 - m + i0} \times \\ \times (+ie\gamma^\mu) \times \frac{i}{\not{q}_3 - m + i0} \times (+ie\gamma^\nu) \times \frac{i}{\not{q}_4 - m + i0}\Big]. \quad (61)$$

Note that a trace of a matrix product depends only on the cyclic order of the matrices, $\mathrm{tr}(ABC\cdots YZ) = \mathrm{tr}(BC\cdots YZA) = \mathrm{tr}(C\cdots YZAB) = \cdots = \mathrm{tr}(ZABC\cdots Y)$. Thus, in eq. (61), we may start the product with any vertex or propagator — as long as we multiply them all in the correct *cyclic order*, the trace will be the same.

As to the Lorentz vector indices $\lambda, \mu, \nu, \ldots$, the index of a vertex should be contracted to the index of the photonic line connected to that vertex. For example, the following diagram for the $e^- + e^- \to e^- + e^-$ scattering



(62)

evaluates to

$$i\mathcal{M} \;=\; \Big(\bar{u}(p_1', s_1') \times (+ie\gamma_\mu) \times u(p_1, s_1)\Big) \times \Big(\bar{u}(p_2', s_2') \times (+ie\gamma_\nu) \times u(p_2, s_2)\Big) \times \frac{-ig^{\mu\nu}}{q^2}\,. \quad (63)$$

Here I have used the Feynman gauge for the photon propagator, but any other gauge would produce exactly the same amplitude

$$\begin{aligned} i\mathcal{M} \;&=\; \bar{u}_1'(ie\gamma_\mu)u_1 \times \bar{u}_2'(ie\gamma_\nu)u_2 \times \frac{i(-g^{\mu\nu} \;+\; c^\mu q^\nu \;+\; q^\mu c^\nu)}{q^2} \\ &=\; \bar{u}_1'(ie\gamma_\mu)u_1 \times \bar{u}_2'(ie\gamma_\nu)u_2 \times \frac{-ig^{\mu\nu}}{q^2} \end{aligned} \quad (64)$$

because of *Gordon identities*

$$\bar{u}_1'(ie\gamma_\mu)u_1 \times q^\mu \;=\; \bar{u}_2'(ie\gamma_\nu)u_2 \times q^\nu \;=\; 0. \quad (65)$$

To prove these identities, we note that the spinors $u_1 \equiv u(p_1, s_1)$ and $\bar{u}_1' \equiv \bar{u}(p_1', s_1')$ obey the Dirac equations for the appropriate momenta,

$$\not{p}_1 u_1 \;=\; m u_1\,, \qquad \bar{u}_1' \not{p}_1' \;=\; m\bar{u}_1'\,. \quad (66)$$

Moreover, $q = p_1' - p_1$ and hence

$$\bar{u}_1'\gamma_\mu u_1 \times q^\mu \;=\; \bar{u}_1' \not{q} u_1 \;=\; \bar{u}_1'(\not{p}_1' - \not{p}_1)u_1 \;=\; (m\bar{u}_1')u_1 \;-\; \bar{u}_1'(mu_1) \;=\; 0. \quad (67)$$

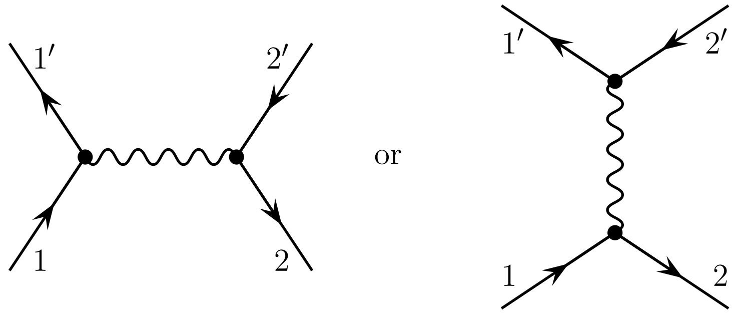
Similarly, $q = p_2 - p_2'$ and hence

$$\bar{u}_2'\gamma_\mu u_2 \times q^\mu \;=\; \bar{u}_2' \not{q} u_2 \;=\; \bar{u}_1'(\not{p}_2 - \not{p}_2')u_1 \;=\; \bar{u}_2'(mu_2)u_2 \;-\; (m\bar{u}_2') \;=\; 0. \quad (68)$$

For other combinations of incoming or outgoing electrons or positrons connected to the same vertex we have similar Gordon identities:

$$\begin{aligned}
\bar{v}(p)\,\not{q}v(p') &= 0 \quad \text{for } q = p' - p, \\
\bar{v}(p_2)\,\not{q}u(p_1) &= 0 \quad \text{for } q = p_1 + p_2\,, \\
\bar{u}(p_2')\,\not{q}v(p_2') &= 0 \quad \text{for } q = p_1' + p_2'\,,
\end{aligned} \tag{69}$$

which provide for gauge invariance of the Feynman diagrams like

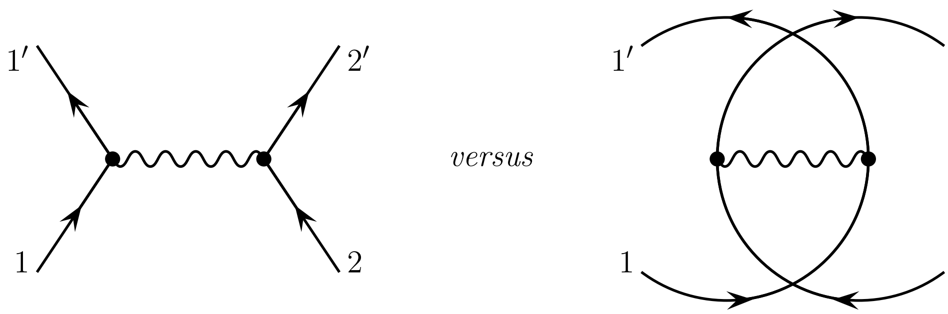


(70)

In general, an individual Feynman diagram is not always gauge-independent. However, when one sums over all diagrams contributing to some scattering process at some order, the sum is always gauge invariant. We shall return to this issue later this semester.

To complete the QED Feynman rules, we need to keep track of the '$-$' signs arising from re-ordering of the fermionic fields and creation / annihilation operators. To save time, I will not go through the gory details of the perturbation theory. Instead, let me simply state the rules for the overall sign of a Feynman diagram in terms of the continuous fermionic lines:

- There is a '$-$' sign for every closed fermionic loop.
- There is a '$-$' sign for every open fermionic line which begins at an outgoing positron and ends at an incoming positron.
- There is a '$-$' sign for every crossing of the fermionic lines.
  Although the number of such crossing depends on how we draw the diagram on a 2D sheet of paper, for example



(71)

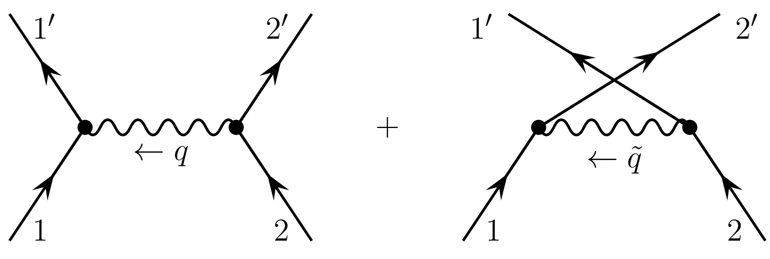
its *parity* #crossings **mod 2** is a topological invariant, and that's all we need to determine the overall sign of the diagram.

⋆ If multiple Feynman diagrams contribute to the same process, then the external legs should stick out from the diagram in the same order for all the diagrams. Or at least all the fermionic external legs should stick out in the same order, which should also agree with the order of fermions in the bra and ket states of the S–matrix element $\langle e^{-\prime},\ldots,e^{-\prime},e^{+\prime},\ldots,e^{+\prime},\gamma',\ldots,\gamma' \big| \mathcal{M} \big| e^-,\ldots,e^-,e^+,\ldots,e^+,\gamma,\ldots,\gamma\rangle$ for the process in question.

Finally, the QED is usually extended to include other charged fermions besides $e^\mp$. The simplest extension includes the muons $\mu^\mp$ and the tau leptons $\tau^\mp$ which behave exactly like the electrons, except for larger masses: while $m_e = 0.51100$ MeV, $m_\mu = 105.66$ MeV and $m_\tau = 1777$ MeV. In terms of the Feynman rules, the muons and the taus have exactly the same vertices, propagators, or external line factors as the electrons, except for a different mass m in the propagators. To distinguish between the 3 lepton species, one should label the solid lines with e, μ, or τ. Different species do not mix, so a label belongs to the whole continuous fermionic line; for an open line, the species must agree with the incoming / outgoing particles at the ends of the line; for a closed loop, one should sum over the species $\ell = e, \mu, \tau$.

Coulomb Scattering

As an example of QED Feynman rules, consider the elastic scattering of two electrons, $e^- + e^- \to e^- + e^-$. There are two tree diagrams contributing to this process,



(72)

which are related by exchanging the final-state electrons, $1' \leftrightarrow 2'$. The first diagram (72) was evaluated back in eq. (63) as

$$i\mathcal{M}_1 \;=\; \frac{-ig^{\mu\nu}}{q^2=t}\times\bar u_1'(ie\gamma_\mu)u_1\times\bar u_2'(ie\gamma_\nu)u_2. \tag{73}$$

For the second diagram, we exchange $u_1'\leftrightarrow u_2'$, change the momentum of the virtual photon from $q=p_1'-p_1$ to $\tilde q=p_2'-p_1$ and hence $q^2=t$ in the denominator to $\tilde q^2=u$, and there is an overall minus sign due to electron line crossing, thus

$$i\mathcal{M}_2 \;=\; -\frac{-ig^{\mu\nu}}{\tilde q^2=u}\times\bar u_2'(ie\gamma_\mu)u_1\times\bar u_1'(ie\gamma_\nu)u_2. \tag{74}$$

Combining the two diagrams, we obtain the net tree-level scattering amplitude as

$$\mathcal{M}_{\rm tree} \;=\; \mathcal{M}_1\;+\;\mathcal{M}_2\;=\;\frac{e^2}{t}\times\bar u_1'\gamma_\mu u_1\times\bar u_2'\gamma^\mu u_2-\;\frac{e^2}{u}\times\bar u_2'\gamma_\mu u_1\times\bar u_1'\gamma^\mu u_2\,. \tag{75}$$

Now let's take the non-relativistic limit of this scattering amplitude. A non-relativistic electron with 3-momentum $|\mathbf{p}|\ll m$ and energy $p^0\approx m$ has plane-wave Dirac spinor

$$u(p,s)\;\approx\;\begin{pmatrix}\sqrt{m}\,\xi\\ \sqrt{m}\,\xi\end{pmatrix}\;+O(|\mathbf{p}|/\sqrt{m}). \tag{76}$$

Consequently, the Dirac sandwiches $\bar u'\gamma^\mu u$ between non-relativistic electron spinors are approximately

$$\begin{aligned}\bar u(p',s')\gamma^0u(p,s)\;&=\;u^\dagger(p',s')u(p,s)\;\approx\;2m\times\xi^\dagger_{s'}\xi_s\;=\;2m\times\delta_{s,s'}\,,\\ \bar u(p',s')\vec\gamma u(p,s)\;&=\;u^\dagger(p',s')\begin{pmatrix}+\vec\sigma & 0\\ 0 & -\vec\sigma\end{pmatrix}u(p,s)\;=\;0\times m\;+\;O(\mathbf{p},\mathbf{p}')\;\ll\;m,\end{aligned} \tag{77}$$

so the scattering amplitude (75) is dominated by the $\mu=0$ terms. Specifically,

$$\begin{aligned}\mathcal{M}^{\rm non.rel.}_{\rm tree}\;&\approx\;\frac{e^2}{t}\times 2m\delta_{s_1',s_1}\times 2m\delta_{s_2',s_2}\;-\;\frac{e^2}{u}\times 2m\delta_{s_2',s_1}\times 2m\delta_{s_1',s_2}\\ &=\;-\frac{4m^2e^2}{\mathbf{q}^2}\times\delta_{s_1',s_1}\delta_{s_2',s_2}\;+\;\frac{4m^2e^2}{\tilde{\mathbf{q}}^2}\times\delta_{s_2',s_1}\delta_{s_1',s_2}\,,\end{aligned} \tag{78}$$

where on the second line I have used $t\approx-(\mathbf{q}=\mathbf{p}_1'-\mathbf{p}_1)^2$ and $u\approx-(\tilde{\mathbf{q}}=\mathbf{p}_2'-\mathbf{p}_1)^2$ for the non-relativistic electrons (since $q_0=E_1'-E_1=O(\mathbf{p}^2/m)\ll|\mathbf{q}|$ and likewise $\tilde q_0\ll|\tilde{\mathbf{q}}|$).

Note that despite the non-relativistic limit, the amplitude (78) is relativistically normalized. In terms of the non-relativistically normalized scattering amplitude

$$f \;=\; \frac{\mathcal{M}}{8\pi E_{\rm cm}} \tag{79}$$

where $E_{\rm cm} \approx 2m$, we have

$$f_{\rm tree}^{\rm non.rel.} \;\approx\; -\frac{me^2}{4\pi\mathbf{q}^2} \times \delta_{s_1',s_1}\delta_{s_2',s_2} \;+\; \frac{me^2}{4\pi\tilde{\mathbf{q}}^2} \times \delta_{s_2',s_1}\delta_{s_1',s_2}\,. \tag{80}$$

Now let's compare the QED amplitude (80) to the non-relativistic amplitude in potential scattering. In the non-relativistic limit, the perturbative expansion of QED in powers of e^2 roughly corresponds to the Born series in potential scattering, so the tree-level amplitude (78) should be compared to the first Born approximation

$$f_B(\mathbf{p} \to \mathbf{p}') \;=\; -\frac{M_{\rm red}}{2\pi} \times \widetilde{V}(\mathbf{q} = \mathbf{p}_1' - \mathbf{p}_1) \;=\; -\frac{M_{\rm red}}{2\pi} \times \int\! d^3\mathbf{x}_{\rm rel}\, V(\mathbf{x}_{\rm rel}) e^{-i\mathbf{q}\cdot\mathbf{x}_{\rm rel}}. \tag{81}$$

Or rather, this is the Born amplitude for distinct spinless particles with a reduced mass $M_{\rm red}$. For particles with spin but interacting with a spin-blind potential $V(\mathbf{x}_1 - \mathbf{x}_2)$, the Born amplitude is

$$f_B \;=\; -\frac{M_{\rm red}}{2\pi} \times \widetilde{V}(\mathbf{q}) \times \delta_{s_1',s_1}\delta_{s_2',s_2} \tag{82}$$

if the two particles are distinct, and if they are identical fermions then there are two such terms related by particle permutation,

$$f_B \;=\; -\frac{M_{\rm red} = \frac{1}{2}m}{2\pi} \times \widetilde{V}(\mathbf{q}) \times \delta_{s_1',s_1}\delta_{s_2',s_2} \;+\; \frac{M_{\rm red} = \frac{1}{2}m}{2\pi} \times \widetilde{V}(\tilde{\mathbf{q}}) \times \delta_{s_2',s_1}\delta_{s_1',s_2}\,. \tag{83}$$

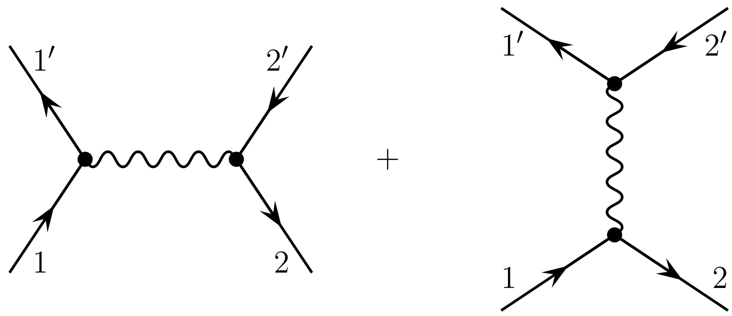
Comparing this Born amplitude to the non-relativistic limit of the tree-level QED amplitude (80), we immediately see that the QED amplitude is a special case of the Born amplitude for

$$\widetilde{V}(\mathbf{q}) \;=\; +\frac{e^2}{\mathbf{q}^2}\,. \tag{84}$$

Fourier transforming this formula back to the coordinate space gives us the good old Coulomb potential for the two electrons,

$$V(\mathbf{x}_1 - \mathbf{x}_2) \;=\; \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, e^{i(\mathbf{x}_1-\mathbf{x}_2)\cdot\mathbf{q}} \times \frac{+e^2}{\mathbf{q}^2} \;=\; \frac{+e^2}{4\pi\,|\mathbf{x}_1 - \mathbf{x}_2|}\,. \tag{85}$$

Now consider the electron-positron elastic scattering $e^- + e^+ \to e^- + e^+$ and its non-relativistic limit. Again, there are two tree diagrams contributing to this process



(86)

which evaluate to

$$i\mathcal{M}_{\rm tree} \;=\; -\bar{u}_1'(ie\gamma_\mu)u_1 \times \bar{v}_2(ie\gamma_\nu)v_2' \times \frac{-ig^{\mu\nu}}{t} \;+\; \bar{v}_2(ie\gamma_\mu)u_1 \times \bar{u}_1'(ie\gamma_\nu)v_2' \times \frac{-ig^{\mu\nu}}{s}\,. \tag{87}$$

The overall minus sign of the first term here is due to the outgoing-positron-to-incoming-positron line in the first diagram; in the second diagram, the incoming electron-to-incoming-positron or the outgoing-positron-to-outgoing-electron lines do not carry minus signs. This time, there is no symmetry between the two diagrams (86), and the corresponding amplitudes have rather different non-relativistic limits. In particular, the denominator of the first diagram becomes $t \approx -\mathbf{q}^2 \ll m^2$ (in absolute value) while the second diagram's denominator has a much larger value $s \approx (2m)^2$. Consequently, the non-relativistic electron-positron scattering is dominated by the t-channel diagram, thus

$$\mathcal{M}_{\rm tree}^{\rm non.rel.} \;\approx\; -\frac{e^2}{t} \times \bar{u}_1'\gamma_\mu u_1 \times \bar{v}_2\gamma^\mu v_2'\,. \tag{88}$$

Moreover, in the non-relativistic limit

$$\bar{u}_1'\gamma^0 u_1 \;\approx\; +2m \times \delta_{s_1',s_1}\,, \quad \bar{v}_2\gamma^0 v_2' \;\approx\; +2m \times \delta_{s_2',s_2}\,, \tag{89}$$

while for $\mu \neq 0$

$$\bar{u}_1' \vec{\gamma} u_1\,,\ \bar{v}_2 \vec{\gamma} v_2' \ =\ O(\mathbf{p})\ \ll\ m, \tag{90}$$

hence

$$\mathcal{M}^{\text{non.rel.}}_{\text{tree}}\ \approx\ -\frac{e^2}{t} \times 2m\delta_{s_1',s_1} \times 2m\delta_{s_2',s_2}\ \approx\ +\frac{4m^2e^2}{\mathbf{q}^2} \times \delta_{s_1',s_1}\delta_{s_2',s_2}\,, \tag{91}$$

or in the non-relativistic normalization

$$f^{\text{non.rel.}}_{\text{tree}}\ \approx\ +\frac{me^2}{4\pi\mathbf{q}^2} \times \delta_{s_1',s_1}\delta_{s_2',s_2}\,. \tag{92}$$

Comparing this QED amplitude to the Born amplitude (82) for *distinct* fermions (since an e^- is distinct from an e^+), we see that they agree for

$$\widetilde{V}(\mathbf{q})\ =\ -\frac{e^2}{\mathbf{q}^2}\,. \tag{93}$$

In coordinate space terms, this means the attractive Coulomb potential

$$V(\mathbf{x}_1 - \mathbf{x}_2)\ =\ \frac{-e^2}{4\pi\,|\mathbf{x}_1 - \mathbf{x}_2|}\,. \tag{94}$$

Thus QED perturbation theory confirms the oldest law of electrostatics: *the like-sign charges repel, while the unlike-sign charges attract.*

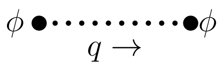
However, this rule works only for the forces arising from exchanges of virtual odd spin bosons — such as photons. The forces arising from exchanges of virtual even spin bosons — such as scalar mesons, or gravitons — do not change sign when one of the two particles is replaced with its anti-particle. Thus, the gravity force is always attractive. Likewise, the Yukawa force due to an isoscalar scalar meson is attractive for all combinations of nucleons and antinucleons — NN, $\overline{NN}$, or $N\overline{N}$.

Yukawa Potential

The Yukawa theory and the Yukawa potential are discussed in detail in §4.7 of the *Peskin and Schroeder* textbook, so in these notes let me simply highlight the differences between the Yukawa theory and the QED. Instead of the EM field, the Yukawa theory has a scalar field ϕ, thus

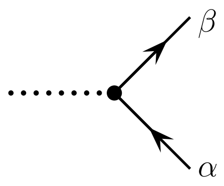
$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial_\mu\phi)^2 \;-\; \tfrac{1}{2}m^2\phi^2 \;+\; \overline{\Psi}(i\partial\!\!\!/ - M)\Psi \;-\; g\phi\times\overline{\Psi}\Psi. \tag{95}$$

(For simplicity, I assume a single fermion species.) The Feynman rules of the Yukawa theory have the same fermionic propagators, external line factors, and sign rules as QED, but instead of photon propagators it has scalar propagators — drawn as dotted lines



$$= \frac{+i}{q^2 - m^2 + i0}\,, \tag{96}$$

and instead of the fermion-antifermion-photon vertices of QED the Yukawa theory has fermion-antifermion-scalar vertices



$$= -ig\delta_{\beta\alpha}\,. \tag{97}$$

without the γ^μ matrices.

Consequently, evaluating the scalar analogues of the diagrams (72) and (86) for the $ff \to ff$ and $f\bar{f} \to f\bar{f}$ scattering processes, we obtain

$$\begin{aligned} \mathcal{M}(ff\to ff) \;&=\; -\frac{g^2}{t-m^2}\times\bar{u}_1'u_1\times\bar{u}_2'u_2 \;+\; \frac{g^2}{u-m^2}\times\bar{u}_2'u_1\times\bar{u}_1'u_2\,. \\ \mathcal{M}(f\bar{f}\to f\bar{f}) \;&=\; +\frac{g^2}{t-m^2}\times\bar{u}_1'u_1\times\bar{v}_2v_2' \;-\; \frac{g^2}{s-m^2}\times\bar{v}_2u_1\times\bar{u}_2'v_2'\,. \end{aligned} \tag{98}$$

In the non-relativistic limit

$$\bar{u}(p',s')u(p,s) \;\approx\; +2M\delta_{s',s} \quad\text{but}\quad \bar{v}(p,s)v(p',s') \;\approx\; -2M\delta_{s',s}\,, \tag{99}$$

while

$$\bar{v}(p_2,s_2)u(p_1,s_1),\ \bar{u}(p_1',s_1')v(p_2',s_2')\ =\ O(\mathbf{p})\ \ll\ M, \tag{100}$$

hence

$$\begin{aligned}\mathcal{M}(ff\to ff)\ &\approx\ -\frac{4M^2g^2}{t-m^2}\times\delta_{s_1',s_1}\delta_{s_2',s_2}\ +\ \frac{4M^2g^2}{u-m^2}\times\delta_{s_2',s_1}\delta_{s_1',s_2}\,,\\ \mathcal{M}(f\bar{f}\to f\bar{f})\ &\approx\ -\frac{4M^2g^2}{t-m^2}\times\delta_{s_1',s_1}\delta_{s_2',s_2}\ +\ \frac{g^2}{s-m^2}\times O(\mathbf{p}^2).\end{aligned} \tag{101}$$

Assuming the scalar is much lighter than the fermion, $m\ll M$, and taking the fermions' 3-momenta $\mathbf{p},\mathbf{p}'$ to be $O(m)\ll M$, we have

$$\frac{1}{t-m^2}\ \approx\ \frac{-1}{\mathbf{q}^2+m^2}\ \gg\ \frac{1}{M^2}\,,\quad \frac{1}{u-m^2}\ \approx\ \frac{-1}{\tilde{\mathbf{q}}^2+m^2}\ \gg\ \frac{1}{M^2}\,,\quad \text{but}\quad \frac{1}{s-m^2}\ \approx\ \frac{+1}{4M^2}\,, \tag{102}$$

hence

$$\begin{aligned}\mathcal{M}(ff\to ff)\ &\approx\ +\frac{4M^2g^2}{\mathbf{q}^2+m^2}\times\delta_{s_1',s_1}\delta_{s_2',s_2}\ -\ \frac{4M^2g^2}{\tilde{\mathbf{q}}^2+m^2}\times\delta_{s_2',s_1}\delta_{s_1',s_2}\,,\\ \mathcal{M}(f\bar{f}\to f\bar{f})\ &\approx\ +\frac{4M^2g^2}{\mathbf{q}^2+m^2}\times\delta_{s_1',s_1}\delta_{s_2',s_2}\ +\ 0.\end{aligned} \tag{103}$$

In the non-relativistic normalization, these amplitudes become

$$\begin{aligned}f(ff\to ff)\ &\approx\ +\frac{Mg^2}{4\pi(\mathbf{q}^2+m^2)}\times\delta_{s_1',s_1}\delta_{s_2',s_2}\ -\ \frac{Mg^2}{4\pi(\tilde{\mathbf{q}}^2+m^2)}\times\delta_{s_2',s_1}\delta_{s_1',s_2}\,,\\ f(f\bar{f}\to f\bar{f})\ &\approx\ +\frac{Mg^2}{4\pi(\mathbf{q}^2+m^2)}\times\delta_{s_1',s_1}\delta_{s_2',s_2}\,,\end{aligned} \tag{104}$$

which agree with Born amplitudes for

$$\widetilde{V}_{ff}(\mathbf{q})\ =\ \widetilde{V}_{f\bar{f}}\ =\ -\frac{g^2}{\mathbf{q}^2+m^2}\,. \tag{105}$$

Fourier transforming this formula back too coordinate space gives us the Yukawa potential

$$V_{ff}(\mathbf{x}_1-\mathbf{x}_2)\ =\ V_{f\bar{f}}(\mathbf{x}_1-\mathbf{x}_2)\ =\ -\frac{g^2}{4\pi}\times\frac{e^{-mr}}{r}\quad\text{for}\quad r\ =\ |\mathbf{x}_1-\mathbf{x}_2|\,. \tag{106}$$

Note the signs of these potentials — two fermions attract each other, and a fermion and an antifermion also attract each other!