Ward Identities

Consider a QED scattering process in which a photon is emitted,

$$X \;\to\; Y \;+\; \gamma, \tag{1}$$

where X and Y are some combinations of electrons and positrons, never mind the details. Any Feynman diagram contributing to such process has an outgoing photon line, hence

$$\mathcal{M}(X \to Y + \gamma) \;=\; \mathcal{E}^*_\mu(k,\lambda) \times \mathcal{M}^\mu \tag{2}$$

where $\mathcal{M}^\mu$ comprises the rest of the diagram — the vertices, the internal lines, and the external lines for the electrons and positrons. If there are multiple diagrams for the process, they all contain the $\mathcal{E}^*_\mu(k,\lambda)$ factor, so the whole amplitude has form (2). The net $\mathcal{M}^\mu$ factor depends on the momenta of all the particles and on the spin states of the electrons and positrons, but it does not depend on the outgoing photon's polarization — that dependence is carried by the $\mathcal{E}^*_\mu$ factor.

Likewise, a process

$$X \;+\; \gamma \;\to\; Y, \tag{3}$$

in which a photon is absorbed has amplitude of the form

$$\mathcal{M}(X + \gamma \to Y) \;=\; \mathcal{E}_\mu(k,\lambda) \times \mathcal{M}^\mu \tag{4}$$

where $\mathcal{M}^\mu$ depends on all the momenta and also electron/positron spins, but not on the incoming photon's polarization.

As Lorentz vectors, the factors $\mathcal{M}^\mu(k,\ldots)$ are $\perp$ to the photon's momentum k^μ,

$$k_\mu \times \mathcal{M}^\mu(k,\ldots) \;=\; 0. \tag{5}$$

This equation — and similar formulae for amplitudes involving multiple photons — are called the *Ward Identities* after John Clive Ward who derived them in 1950. Physically, they stem from the electric current conservation, $\partial_\mu J^\mu(x) = 0$, or in momentum space $k_\mu \times J^\mu(k) = 0$. Next semester I shall prove the Ward identities in painful detail, but right now let me simply show you what these identities are good for.

First of all, the Ward identity (5) provides for the gauge invariance of the scattering amplitudes (2) and (4) despite the gauge-dependence of the photonic polarization vectors $\mathcal{E}_\mu(k,\lambda)$. Indeed, different gauge conditions for the quantum EM potential fields

$$\hat{A}^\mu(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega}\sum_{\lambda=\pm1}\Big(e^{-ikx}\mathcal{E}^\mu(k,\lambda)\hat{a}_{k,\lambda}\;+\;e^{+ikx}\mathcal{E}^{\mu*}(k,\lambda)\hat{a}^\dagger_{k,\lambda}\Big) \tag{6}$$

lead to different polarization vectors $\mathcal{E}_\mu(k,\lambda)$, but since

$$\hat{A}^\mu(x)[\text{in one gauge}] \;-\; \hat{A}^\mu(x)[\text{in another gauge}] \;=\; \partial^\mu\Big(\text{some }\hat{\Lambda}(x)\Big), \tag{7}$$

we must have

$$\mathcal{E}^\mu(k,\lambda)[\text{in one gauge}] \;-\; \mathcal{E}^\mu(k,\lambda)[\text{in another gauge}] \;=\; k^\mu\times\Big(\text{some }f(k,\lambda)\Big). \tag{8}$$

Consequently, the Ward identity $k^\mu\mathcal{M}_\mu = 0$ provides for the gauge-invariance of the amplitudes $\mathcal{M} = \mathcal{E}^\mu W_\mu$ and $\mathcal{M} = \mathcal{E}^{\mu*}\mathcal{M}_\mu$.

For processes involving two external photons we have similar formulae:

$$\begin{aligned}
\mathcal{M}(X\to Y+\gamma+\gamma) \;&=\; \mathcal{E}^*_\mu(k_1,\lambda_1)\mathcal{E}^*_\nu(k_2,\lambda_2)\times\mathcal{M}^{\mu\nu}(k_1,k_2,\ldots),\\
\mathcal{M}(X+\gamma\to Y+\gamma) \;&=\; \mathcal{E}_\mu(k_1,\lambda_1)\mathcal{E}^*_\nu(k_2,\lambda_2)\times\mathcal{M}^{\mu\nu}(k_1,k_2,\ldots),\\
\mathcal{M}(X+\gamma+\gamma\to Y) \;&=\; \mathcal{E}_\mu(k_1,\lambda_1)\mathcal{E}_\nu(k_2,\lambda_2)\times\mathcal{M}^{\mu\nu}(k_1,k_2,\ldots),
\end{aligned} \tag{9}$$

and in all 3 cases the $\mathcal{M}^{\mu\nu}$ obeys two Ward identities,

$$k_{1\mu}\times\mathcal{M}^{\mu\nu}(k_1,k_2,\ldots) \;=\; 0 \quad\text{and}\quad k_{2\nu}\times\mathcal{M}^{\mu\nu}(k_1,k_2,\ldots) \;=\; 0, \tag{10}$$

one identity for each photon. Again, these identities provide for the gauge invariance of the scattering amplitudes (9).

Generalization to processes involving more photons is completely straightforward. A process involving N external photons — incoming or outgoing — has amplitude of the form

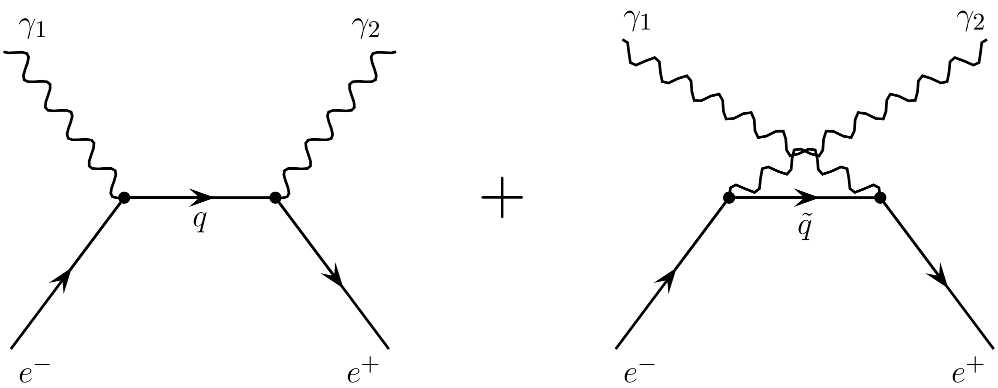
$$\mathcal{M}(X+\text{photons} \to Y+\text{photons}) \;=\; \mathcal{E}^*_{\mu_1}(k_1,\lambda_1)\times\cdots\times\mathcal{E}_{\mu_N}(k_N,\lambda_N)\times\mathcal{M}^{\mu_1,\cdots\mu_N}(k_1,\ldots,k_n,\ldots), \tag{11}$$

where the last factors obeys Ward identities for each of the photons,

$$\forall i=1,\ldots,N,\quad k_{i,\mu_i}\times\mathcal{M}^{\mu_1,\cdots\mu_N}(k_1,\ldots,k_n,\ldots) \;=\; 0. \tag{12}$$

CAVEAT:

The Ward identities do not work for separate Feynman diagrams, but only to complete sums of all the diagrams that contribute to a particular process at each order of the perturbation theory. For example, consider the annihilation process $e^+ + e^- \to 2\gamma$. At the three level, there are two diagrams contributing to the annihilation amplitude



(13)

thus

$$\mathcal{M} \;=\; \mathcal{E}^*_\mu(k_1,\lambda_1)\mathcal{E}^*_\nu(k_2,\lambda_2)\times\mathcal{M}^{\mu\nu} \quad\text{for}\quad \mathcal{M}^{\mu\nu} \;=\; \mathcal{M}^{\mu\nu}_1 \;+\; \mathcal{M}^{\mu\nu}_2 \tag{14}$$

where $\mathcal{M}^{\mu\nu}_1$ comes from the first diagram and $\mathcal{M}^{\mu\nu}_2$ from the second diagram. These amplitudes are evaluated in detail in the next set of my notes, where we shall see that the $\mathcal{M}^{\mu\nu}_1$ and the $\mathcal{M}^{\mu\nu}_2$ do not obey the Ward identities by themselves, but only after we add them together,

$$k_{1\mu}\times\mathcal{M}^{\mu\nu}_1 \;\neq\; 0,\quad k_{1\mu}\times\mathcal{M}^{\mu\nu}_2 \;\neq\; 0,\quad\text{but}\quad k_{1\mu}\times\mathcal{M}^{\mu\nu}_{\text{net}} \;=\; 0, \tag{15}$$

and likewise for the second photon.

Polarization Sums

Besides proving gauge invariance of the photonic amplitudes, the Ward identities are good for summing amplitudes[2] over photonic polarizations. For example, for the $X \to Y + \gamma$ processes involving a single outgoing photon, the Ward identities lead to

$$\overline{|\mathcal{M}|^2} \stackrel{\text{def}}{=} \sum_\lambda \left|\mathcal{E}^*_\mu(k,\lambda) \times \mathcal{M}^\mu\right|^2 = -\mathcal{M}^\mu \mathcal{M}^*_\mu \equiv -g_{\mu\nu}\mathcal{M}^\mu\mathcal{M}^{\nu *}. \tag{16}$$

To simplify the proof of this relation, let me work in the coordinate frame where the photon moves in the x^3 direction, thus $k^\mu = (\omega, 0, 0, \omega)$. In this frame, the Ward identity becomes

$$k_\mu \times \mathcal{M}^\mu = \omega\mathcal{M}^0 - \omega\mathcal{M}^3 = 0 \quad\Longrightarrow\quad \mathcal{M}^0 = \mathcal{M}^3. \tag{17}$$

The polarization vectors for the photon helicities $\lambda = +1$ and $\lambda = -1$ and $k^\mu = (\omega, 0, 0, \omega)$ are respectively

$$\mathcal{E}^\mu(\lambda = +1) = \left(c_R, \tfrac{+1}{\sqrt{2}}, \tfrac{-i}{\sqrt{2}}, c_R\right) \quad \text{and} \quad \mathcal{E}^\mu(\lambda = -1) = \left(c_L, \tfrac{-1}{\sqrt{2}}, \tfrac{-i}{\sqrt{2}}, c_L\right) \tag{18}$$

where c_L and c_R depends on the gauge (in the Coulomb gauge $c_L = c_R = 0$), but thanks to the Ward identity (17) this gauge dependence cancels out from the polarized scattering amplitudes

$$\begin{aligned}
\mathcal{M}(\gamma_L) &= \mathcal{E}^*_\mu(\lambda = -1) \times \mathcal{M}^\mu = c_L \times \left(\mathcal{M}^0 - \mathcal{M}^3 = 0\right) - \frac{-\mathcal{M}^1 + i\mathcal{M}^2}{\sqrt{2}}, \\
&= \frac{+\mathcal{M}^1 - i\mathcal{M}^2}{\sqrt{2}},
\end{aligned} \tag{19}$$

$$\begin{aligned}
\mathcal{M}(\gamma_R) &= \mathcal{E}^*_\mu(\lambda = +1) \times \mathcal{M}^\mu = c_R \times \left(\mathcal{M}^0 - \mathcal{M}^3 = 0\right) - \frac{+\mathcal{M}^1 + i\mathcal{M}^2}{\sqrt{2}}. \\
&= \frac{-\mathcal{M}^1 - i\mathcal{M}^2}{\sqrt{2}}.
\end{aligned} \tag{20}$$

Now suppose the photon detector is un-polarized, *i.e.*, equally sensitive to both photon polarizations. Summing the $|\mathcal{M}|^2$ over $\lambda = \pm 1$, we obtain

$$\overline{|\mathcal{M}|^2} = |\mathcal{M}(\gamma_L)|^2 + |\mathcal{M}(\gamma_R)|^2 = \left|\frac{\mathcal{M}^1 - i\mathcal{M}^2}{\sqrt{2}}\right|^2 + \left|\frac{-\mathcal{M}^1 - i\mathcal{M}^2}{\sqrt{2}}\right|^2 = \left|\mathcal{M}^1\right|^2 + \left|\mathcal{M}^2\right|^2 . \tag{21}$$

Moreover, thanks to the Ward Identity (17), we may re-write the RHS here as

$$\left|\mathcal{M}^1\right|^2 + \left|\mathcal{M}^2\right|^2 = \left|\mathcal{M}^1\right|^2 + \left|\mathcal{M}^2\right|^2 + \cancel{\left|\mathcal{M}^3\right|}^2 - \cancel{\left|\mathcal{M}^0\right|}^2 = -g_{\mu\nu}\mathcal{M}^\mu\mathcal{M}^{*\nu}, \tag{22}$$

hence

$$\overline{|\mathcal{M}|^2} = |\mathcal{M}(\gamma_L)|^2 + |\mathcal{M}(\gamma_R)|^2 = -g_{\mu\nu}\mathcal{M}^\mu\mathcal{M}^{*\nu} \equiv -\mathcal{M}^\mu\mathcal{M}^*_\mu . \tag{16}$$

Although we have derived this formula in a particular coordinate frame (where the photon flies along the z axis), both sides of this equation are Lorentz invariant. Therefore, eq. (16) must be valid in all coordinate frames.

Eq. (16) applies to an outgoing photon whose polarization is not measured, but there is a similar formula for an incoming photon from an un-polarized source

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\Big(|\mathcal{M}(\gamma_L)|^2 + |\mathcal{M}(\gamma_R)|^2\Big) = \frac{1}{2} \times -\mathcal{M}^\mu\mathcal{M}^*_\mu . \tag{23}$$

Likewise, there are straightforward generalizations of eqs. (16) and (23) for processes involving multiple external photons. For example, for two photons

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}(\lambda_1,\lambda_2)|^2 = +\mathcal{M}^{\mu\nu}\mathcal{M}^*_{\mu\nu} , \tag{24}$$

and hence the un-polarized

$$\overline{|\mathcal{M}|^2} = +\mathcal{M}^{\mu\nu}\mathcal{M}^*_{\mu\nu} \times \begin{cases} \frac{1}{4} & \text{for } X + 2\gamma \to Y, \\ \frac{1}{2} & \text{for } X + \gamma \to Y + \gamma, \\ 1 & \text{for } X \to Y + 2\gamma. \end{cases} \tag{25}$$

More generally, for a process involving N external photons — incoming or outgoing — with a polarized amplitude

$$\mathcal{M} \;=\; \mathcal{E}^*_{\mu_1}(k_1,\lambda_1) \times \cdots \times \mathcal{E}_{\mu_N}(k_N,\lambda_N) \times \mathcal{M}^{\mu_1,\dots\mu_N}(k_1,\dots,k_n,\dots), \tag{26}$$

summing $|\mathcal{M}|^2$ over all the photons' polarization results in

$$\sum_{\lambda_1,\dots,\lambda_N} |\mathcal{M}|^2 \;=\; (-1)^N \times \mathcal{M}^{\mu_1,\dots\mu_N} \times \mathcal{M}^*_{\mu_1,\dots\mu_N}\,. \tag{27}$$