CORRELATION FUNCTIONS IN PERTURBATION THEORY

Many aspects of quantum field theory are related to its n-point correlation functions

$$\mathcal{F}_n(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle \tag{1}$$

— or for theories with multiple fields $\hat{\Phi}^a$,

$$\mathcal{F}_n^{a_1,\ldots,a_n}(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H^{a_1}(x_1)\cdots\hat{\Phi}_H^{a_n}(x_n)\,|\Omega\rangle\,. \tag{2}$$

Note that all the fields $\hat{\Phi}_H(x)$ here are in the Heisenberg picture so their time dependence involves the complete Hamiltonian $\hat{H}$ of the interacting theory. Likewise, $|\Omega\rangle$ is the ground state of $\hat{H}$, *i.e.* the true physical vacuum of the theory.

In perturbation theory, the correlation functions $\mathcal{F}_n$ of the interacting theory are related to the free theory's correlation functions

$$\langle 0|\,\mathbf{T}\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\cdots\text{more }\hat{\Phi}_I(z_1)\hat{\Phi}_I(z_2)\cdots|0\rangle\,. \tag{3}$$

involving additional fields $\hat{\Phi}_I(z_1)\hat{\Phi}_I(z_2)\cdots$. Note that in eq. (3) the fields are in the interaction rather than Heisenberg picture, so they evolve with time as free fields according to the free Hamiltonian $\hat{H}_0$. Likewise, $|0\rangle$ is the free theory's vacuum, *i.e.* the ground state of the free Hamiltonian $\hat{H}_0$ rather than the full Hamiltonian $\hat{H}$.

To work out the relation between (1) and (3), we start by formally relating quantum fields in the Heisenberg and the interaction pictures,

$$\hat{\Phi}_H(\mathbf{x},t) \;=\; e^{+i\hat{H}t}\hat{\Phi}_S(\mathbf{x})e^{-i\hat{H}t} \;=\; e^{+i\hat{H}t}e^{-i\hat{H}_0t}\hat{\Phi}_I(\mathbf{x},t)e^{+i\hat{H}_0t}e^{-i\hat{H}t}. \tag{4}$$

We may re-state this relation in terms of evolution operators using a formal expression for the later,

$$\hat{U}_I(t,t_0) \;=\; e^{+i\hat{H}_0t}e^{-i\hat{H}(t-t_0)}e^{-i\hat{H}_0t_0}. \tag{5}$$

Note that this formula applies for both forward and backward evolution, *i.e.* regardless of whether $t > t_0$ or $t < t_0$. In particular,

$$\hat{U}_I(t,0) \;=\; e^{+i\hat{H}_0 t}e^{-i\hat{H}t} \qquad \text{and} \qquad \hat{U}_I(0,t) \;=\; e^{+i\hat{H}t}e^{-i\hat{H}_0 t}, \tag{6}$$

which allows us to re-state eq. (4) as

$$\hat{\Phi}_H(x) \;=\; \hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,0). \tag{7}$$

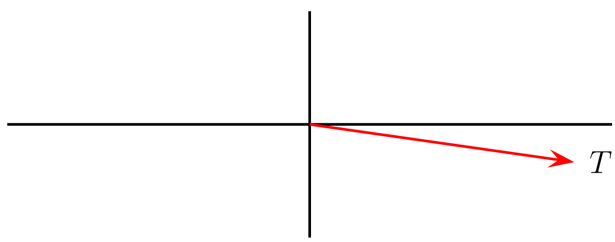
Consequently,

$$\hat{\Phi}_H(x)\hat{\Phi}_H(y) \;=\; \hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,0) \tag{8}$$

because $\hat{U}_I(x^0,0)\hat{U}_I(0,y^0) = \hat{U}_I(x^0,y^0)$, and likewise for n fields

$$\hat{\Phi}_H(x_1)\hat{\Phi}_H(x_2)\cdots\hat{\Phi}_H(x_n) \;= \tag{9}$$
$$=\; \hat{U}_I(0,x_1^0)\hat{\Phi}_I(x_1)\hat{U}_I(x_1^0,x_2^0)\hat{\Phi}_I(x_2)\cdots\hat{U}_I(x_{n-1}^0,x_n^0)\hat{\Phi}_I(x_n)\hat{U}_I(x_n^0,0).$$

Now we need to relate the free vacuum $|0\rangle$ and the true physical vacuum $|\Omega\rangle$. Consider the state $\hat{U}_I(0,-T)\,|0\rangle$ for a complex T, and take the limit of $T \to (+1 - i\epsilon) \times \infty$. That is, $\operatorname{Re} T \to +\infty$, $\operatorname{Im} T \to -\infty$, but the imaginary part grows slower than the real part. Pictorially, in the complex T plane,



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we go infinitely far to the right at an infinitesimally small angle below the real axis.

Without loss of generality we assume the free theory has zero vacuum energy, thus $\hat{H}_0 |0\rangle = 0$ and hence

$$\hat{U}_I(0,-T)\,|0\rangle \;=\; e^{-i\hat{H}T}e^{+i\hat{H}_0T}\,|0\rangle \;=\; e^{-i\hat{H}T}\,|0\rangle\,. \tag{11}$$

From the interacting theory's point of view, $|0\rangle$ is a superposition of eigenstates $|Q\rangle$ of the full Hamiltonian $\hat{H}$,

$$|0\rangle \;=\; \sum_Q |Q\rangle \times \langle Q|0\rangle \quad\Longrightarrow\quad e^{-i\hat{H}T}\,|0\rangle \;=\; \sum_Q |Q\rangle \times e^{-iTE_Q}\,\langle Q|0\rangle \tag{12}$$

For complex T, $|e^{-iTE_Q}| = \exp(+E_Q\,\mathrm{Im}(T))$, so in the $T \to (+1-i\epsilon)\times\infty$ limit, the sum in the second eq. (12) is dominated by the term with the lowest E_Q. Or rather, it's dominated by the lowest-energy eigenstate $|Q_0\rangle$ of $\hat{H}$ with the same quantum numbers as $|0\rangle$ since otherwise, we would have zero overlap $\langle Q_0|0\rangle$. Obviously, this lowest-energy state is the physical vacuum $|\Omega\rangle$, thus

$$\hat{U}_I(0,-T)\,|0\rangle \;\xrightarrow[T\to(+1-i\epsilon)\infty]{}\; |\Omega\rangle \times e^{-iTE_\Omega}\,\langle\Omega|0\rangle \tag{13}$$

and therefore

$$|\Omega\rangle \;=\; \lim_{T\to(+1-i\epsilon)\infty} \hat{U}_I(0,-T)\,|0\rangle \times \frac{e^{+iTE_\Omega}}{\langle\Omega|0\rangle}\,. \tag{14}$$

Likewise,

$$\langle 0|\,U_I(+T,0) \;=\; \langle 0|\,e^{-i\hat{H}T} \;=\; \sum_Q \langle 0|Q\rangle\,e^{-iTE_Q}\,\langle Q| \;\xrightarrow[T\to(1-i\epsilon)\infty]{}\; \langle 0|\Omega\rangle\,e^{-iTE_\Omega} \times \langle\Omega| \tag{15}$$

and therefore

$$\langle\Omega| \;=\; \lim_{T\to(+1-i\epsilon)\infty} \frac{e^{+iTE_\Omega}}{\langle 0|\Omega\rangle} \times \langle 0|\,\hat{U}_I(+T,0)\,. \tag{16}$$

Combining eqs. (8), (14), and (16), we may now express the two-point function as

$$\langle\Omega|\,\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \;=\; \lim_{T\to(+1-i\epsilon)\infty} C(T) \times \langle 0|\,\mathrm{Big\_Product}\,|0\rangle \tag{17}$$

where

$$C(T) \;=\; \frac{e^{2iTE_\Omega}}{|\langle 0|\Omega\rangle|^2} \tag{18}$$

is a just a coefficient, and

$$\begin{aligned}
\text{Big_Product} &= \hat{U}_I(+T,0)\hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,0)\hat{U}_I(0,-T) \\
&= \hat{U}_I(+T,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,-T).
\end{aligned} \tag{19}$$

For $x^0 > y^0$, the last line here is in proper time order, so if we re-order the operators, the time-orderer $\mathbf{T}$ would put them back where they belong. Thus, using $\mathbf{T}$ to keep track of the operator order, we have

$$\begin{aligned}
\text{Big_Product} &= \mathbf{T}\Big(\hat{U}_I(+T,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,-T)\Big) \\
&= \mathbf{T}\Big(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\hat{U}_I(+T,x^0)\hat{U}_I(x^0,y^0)\hat{U}_I(y^0,-T)\Big) \\
&= \mathbf{T}\Big(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\hat{U}_I(+T,-T)\Big) \\
&= \mathbf{T}\left(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\exp\left(\frac{-i\lambda}{24}\int_{-T}^{+T}\!dt\int\! d^3\mathbf{z}\,\hat{\Phi}_I^4(t,\mathbf{z})\right)\right)
\end{aligned} \tag{20}$$

where the last line follows from the Dyson series for the evolution operator

$$U_I(t_f,t_i) \;=\; \mathbf{T}\text{-exp}\left(-i\int_{t_i}^{t_f}\!dt\,\hat{V}_I(t)\right) \;=\; \mathbf{T}\text{-exp}\left(\frac{-i\lambda}{24}\int_{t_i}^{t_f}\!dt\int\! d^3\mathbf{z}\,\hat{\Phi}_I^4(t,\mathbf{z})\right).$$

Altogether, the two-point correlation function becomes

$$\begin{aligned}
\mathcal{F}_2(x,y) &\stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \\
&= \lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\exp\left(\frac{-i\lambda}{24}\int\! d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle\,,
\end{aligned} \tag{21}$$

where the spacetime integral has ranges

$$\int\! d^4z \;\equiv\; \int_{-T}^{+T}\!dz^0\int_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{z}\,. \tag{22}$$

Similarly, the n-point correlation functions can be written as

$$\mathcal{F}_n(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat\Phi_H(x_1)\cdots\hat\Phi_H(x_n)\,|\Omega\rangle$$
$$= \lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\hat\Phi_I(x_1)\cdots\hat\Phi_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat\Phi_I^4(z)\right)\right)|0\rangle\,. \tag{23}$$

Note that the coefficient $C(T)$ is the same for all the correlations functions (for any n); it's related to the vacuum energy shift according to eq. (18). In particular, for $n=0$ the $\mathcal{F}_0=\langle\Omega|\Omega\rangle=1$, but it's also given by eq. (23), hence

$$\lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat\Phi_I^4(z)\right)\right)|0\rangle \;=\; 1. \tag{24}$$

This allows us to eliminate the $C(T)$ factors from eqs. (23) by taking *ratios* of the free-theory correlation functions,

$$\mathcal{F}_n(x_1,\ldots,x_n) \;=\; \lim_T \frac{\langle 0|\,\mathbf{T}\left(\hat\Phi_I(x_1)\cdots\hat\Phi_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat\Phi_I^4(z)\right)\right)|0\rangle}{\langle 0|\,\mathbf{T}\left(\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat\Phi_I^4(z)\right)\right)|0\rangle}\,. \tag{25}$$

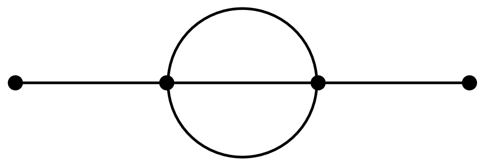
The limit here is $T\to(+1-i\epsilon)\times\infty$, and the T dependence under the limit is implicit in the ranges of the spacetime integrals, *cf.* eq. (22).

In perturbation theory, the vacuum sandwiches in the numerator and the denominator of eq. (25) can be expanded into sums of Feynman diagrams. Indeed, expanding the numerator in a power series in λ, we obtain

$$\langle 0|\mathbf{T}\left(\hat\Phi_I(x_1)\cdots\hat\Phi_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat\Phi_I^4(z)\right)\right)|0\rangle \;=$$
$$= \sum_{N=0}^{\infty}\frac{(-i\lambda)^N}{(4!)^N\,N!}\int\!d^4z_1\cdots\!\int\!d^4z_N\;\langle 0|\,\mathbf{T}\hat\Phi_I(x_1)\cdots\hat\Phi_I(x_n)\times\hat\Phi_I^4(z_1)\cdots\hat\Phi_I^4(z_N)\,|0\rangle \tag{26}$$

where each sub-sandwich $\langle 0|\,\mathbf{T}\hat\Phi_I(x_1)\cdots\hat\Phi_I(x_n)\times\hat\Phi_I^4(z_1)\cdots\hat\Phi_I^4(z_N)\,|0\rangle$ expands into a big sum of products of $\frac{4N+n}{2}$ Feynman propagators $G_F(x_i-x_j)$, $G_F(x_i-z_j)$, or $G_F(z_i-z_j)$. We have gone through expansion back in November — here are my notes — so let me simply summarize the result in terms of the *Feynman rules for the correlation functions*:

⋆ A generic Feynman diagram for the n-point correlation function has n *external vertices* $x_1, \ldots, x_n$ or valence $= 1$ plus some number $N = 0, 1, 2, 3, \ldots$ of *internal vertices* $z_1, \ldots, z_N$ of valence $= 4$. On the other hand, it has no external lines but only the internal lines between the vertices. Here is an example diagram with 2 external vertices, 2 internal vertices, and 5 internal lines:



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- To evaluate a diagram in coordinate space, first multiply the usual factors:
  - ∗ The free propagator $G_F(z_i - z_j)$ for a line connecting vertices internal z_i and z_j, and likewise for lines connecting an internal vertex z_i to an external vertex x_j, or two external vertices x_i and x_j.
  - ∗ $(-i\lambda)$ factor for each internal vertex.
  - ∗ The combinatorial factor 1/#symmetries of the diagram (including the trivial symmetry).
- Second, integrate $\int d^4z$ over each internal vertex location; the integration range is as in eq. (22). But do not integrate over the external vertices — their location's $x_1, \ldots, x_n$ are the arguments of the n-point correlation function $\mathcal{F}_n(x_1, \ldots, x_n)$.
- To calculate the numerator of eq. (25) to order $\lambda^{N_{\max}}$, sum over *all* diagrams with n external vertices, $N \leq N_{\max}$ internal vertices, and any pattern of lines respecting the valences of all the vertices.

At this point, we are summing over all kinds of diagrams, connected or disconnected, and even the vacuum bubbles are allowed. However, similar to what we had back in November, the vacuum bubbles can be factored out:

$$\sum(\text{all diagrams}) = \sum \begin{pmatrix} \text{diagrams without} \\ \text{vacuum bubbles} \end{pmatrix} \times \sum \begin{pmatrix} \text{vacuum bubbles} \\ \text{without external vertices} \end{pmatrix}. \tag{28}$$

Moreover, the vacuum bubble factor here is the same for all the free-theory vacuum sandwiches

$$\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle$$

in the numerators of eqs. (25) for all the correlation functions, and also in the $n=0$ sandwich

$$\langle 0|\,\mathbf{T}\left(\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle \;=\; \sum\left(\begin{matrix}\text{vacuum bubbles}\\ \text{without external vertices}\end{matrix}\right) \tag{29}$$

in the all the denominators. This means that the vacuum bubbles simply cancel out from the correlation functions! In other words,

$$\mathcal{F}_n(x_1,\ldots,x_n) \;=\; \sum\left(\begin{matrix}\text{Feynman diagrams with}\\ n\text{ external vertices } x_1,\ldots x_n\\ \text{and without vacuum bubbles}\end{matrix}\right). \tag{30}$$

Besides reducing the number of diagrams we need to calculate, the cancellation of the vacuum bubbles leads to another simplification: Instead of evaluating each diagram for a finite T, taking the ratio of two sums of diagrams, and only then taking the $T\to(+1-i\epsilon)\infty$ limit, we may now take that limit directly for each diagram. In practice, this means integrating each $\int d^4z_i$ over the whole Minkowski spacetime instead of a limited time range from $-T$ to $+T$ as in eq. (22). Consequently, when we Fourier transform the Feynman rules from the coordinate space to the momentum space, we end up with the usual momentum-conservation factors $(2\pi)^4\delta^{(4)}(\pm q_1\pm q_2\pm q_3\pm q_4)$ at each internal vertex instead of something much more complicated.

So here are the *momentum-space Feynman rules for the correlation functions:*

- Since all the lines are internal, assign a variable momentum q_i^μ to each line and specify the direction of this momentum flow (from which vertex to which vertex).

∗ Each line carries a propagator $\dfrac{i}{q^2-m^2+i0}$.

∗ Each external vertex x carries a factor e^{+iqx} or e^{-iqx}, depending on whether the momentum q flows into or out from the vertex.

∗ Each internal vertex carries factor $(-i\lambda) \times (2\pi)^4 \delta^{(4)}(\pm q_1^{\pm} q_2 \pm q_3 \pm q_4)$.

∗ Overall combinatorial factor 1/#symmetries for the whole diagram.

• Multiply all these factors together, then integrate over all the momenta q_i^μ.

For example, the diagram (27) evaluates to

$$
\begin{aligned}
\mathcal{F}_2(x,y) \supset\ & \frac{1}{6}\int\frac{d^4q_1}{(2\pi)^4}\cdots\int\frac{d^4q_5}{(2\pi)^4}\prod_{i=1}^{5}\frac{i}{q_i^2-m^2+i\epsilon}\times e^{-iq_1x}\times e^{+iq_2y}\times \\
& \times(-i\lambda)(2\pi)^4\delta^{(4)}(q_1-q_3-q_4-q_5)\times \\
& \times(-i\lambda)(2\pi)^4\delta^{(4)}(q_3+q_4+q_5-q_2) \\
=\ & \frac{-i\lambda^2}{6}\int\frac{d^4q_1}{(2\pi)^4}\,e^{-iq_1(x-y)}\times\left(\frac{1}{q_1^2-m^2+i\epsilon}\right)^2\times \\
& \times\iint\frac{d^4q_3\,d^4q_4}{(2\pi)^8}\,\frac{1}{q_3^2-m^2+i\epsilon}\times\frac{1}{q_4^2-m^2+i\epsilon}\times \\
& \times\frac{1}{(q_5=q_1-q_3-q_4)^2-m^2+i\epsilon}
\end{aligned}
\tag{31}
$$

Note: as defined in eq. (1), the correlation functions $\mathcal{F}_n(x_1,\ldots,x_n)$ obtain by summing *all* Feynman diagrams without vacuum bubbles, *cf.* eq. (30). Both connected and disconnected diagrams are included, as long as each connected part of a disconnected diagram has some external vertices. However, the disconnected diagrams' contributions can be re-summed in terms of correlation functions of fewer fields. Indeed, let's define the *connected correlation functions*

$$
\mathcal{F}_n^{\rm conn}(x_1,\ldots,x_n) \;=\; \sum\begin{pmatrix}\textit{connected}\text{ Feynman diagrams}\\ \text{with } n \text{ external vertices}\end{pmatrix}. \tag{32}
$$

Then the original $\mathcal{F}_n$ functions can be obtained from these via *cluster expansion:*

$$
\begin{aligned}
\mathcal{F}_2(x,y) \;&=\; \mathcal{F}_2^{\rm conn}(x,y),\\
\mathcal{F}_4(x,y,x,w) \;&=\; \mathcal{F}_4^{\rm conn}(x,y,z,w) \;+\; \mathcal{F}_2^{\rm conn}(x,y)\times\mathcal{F}_2^{\rm conn}(z,w)\\
&\qquad +\; \mathcal{F}_2^{\rm conn}(x,z)\times\mathcal{F}_2^{\rm conn}(y,w) \;+\; \mathcal{F}_2^{\rm conn}(x,w)\times\mathcal{F}_2^{\rm conn}(y,z),\\
\mathcal{F}_6(x,y,x,u,v,w) \;&=\; \mathcal{F}_6^{\rm conn}(x,y,z,u,v,w)\\
&\qquad +\; \Big(\mathcal{F}_2^{\rm conn}(x,y)\times\mathcal{F}_4^{\rm conn}(z,u,v,w) \;+\; \text{permutations}\Big)\\
&\qquad +\; \Big(\mathcal{F}_2^{\rm conn}(x,y)\times\mathcal{F}_2^{\rm conn}(z,u)\times\mathcal{F}_2^{\rm conn}(v,w) \;+\; \text{permutations}\Big),\\
&\textit{etc.},\ \textit{etc.}
\end{aligned}
\tag{33}
$$

The *connected* 4-point, 6-point, *etc.*, correlation functions are related to the scattering amplitudes via the LSZ reduction formula — named after Harry Lehmann, Kurt Symanzik, and Wolfhart Zimmermann, — see $7.2 of the Peskin and Schroeder textbook for the details. I shall explain the LSZ reduction formula later in class; here are my notes on the subject.

The 2–Point Correlation Function

Meanwhile, let us focus on the 2-point correlation function $\mathcal{F}_2(x-y)$, which is related to the renormalization of the particle mass and the strength of the quantum field. Also, let's put aside the perturbation theory — we shall return to it later in these notes — and focus on the analytic features of the two-point function and their relation to the particle spectrum of the quantum theory.

By definition, the $\mathcal{F}_2(x-y)$ for $x^0>y^0$ amounts to

$$
\begin{aligned}
\mathcal{F}_2(x-y) \;&\stackrel{\rm def}{=}\; \langle\Omega|\,\mathbf{T}\hat\Phi_H(x)\hat\Phi_H(y)\,|\Omega\rangle \;=\; \langle\Omega|\,\hat\Phi_H(x)\hat\Phi_H(y)\,|\Omega\rangle\\
&=\sum_{|\Psi\rangle}\langle\Omega|\,\hat\Phi_H(x)\,|\Psi\rangle\times\langle\Psi|\,\hat\Phi_H(y)\,|\Omega\rangle
\end{aligned}
\tag{34}
$$

where the sum is over all the quantum states $|\Psi\rangle$ of the theory. Or rather, over all quantum states which can be created by the action of the quantum field $\hat\Phi(y)$ on the vacuum state. In the free theory, such states would be limited to the one-particle states with different momenta, but the interacting field $\hat\Phi_H(y)$ may also create a three-particle state, or a five-particle state, *etc.*, *etc.* In a more general theory, the quantum states $|\Psi\rangle$ which could be created by the action of some quantum field $\hat{\varphi}(y)$ on the vacuum include all the multi-particle states which have the same net conserved quantum numbers as a single naive quantum of the field $\hat{\varphi}(y)$. For example, in QED, the states $\hat{A}^\mu(y)\,|\Omega\rangle$ created by the EM field acting on the vacuum include the one-photon states, the three-photon states, *etc.*, but also the electron-positron states — including both the un-bound two-particle states and the hydrogen atom-like bound states, — as well as the states including one or more e^-e^+ pairs and several photons. In other words, all the quantum states which can get mixed with a single-photon state by the QED interactions.

For simplicity, let me keep the states $|\Psi\rangle$ in eq. (34) completely generic. As to their quantum numbers, let me separate the net energy-momentum p^μ of all the particles involved from all the other quantum numbers which I'll denote by the lower-case ψ, thus $|\Psi\rangle = |\psi, p^\mu\rangle$. Note: for the single-particle and bound states, the spectrum of ψ is discrete, while for the un-bound multi-particle states the spectrum of ψ is continuous since ψ includes the relative velocities of the several particles. As to the spectrum of the net momentum p^μ, it spans the positive-energy mass shell for the mass which depends on ψ, thus

$$\text{any }\mathbf{p},\quad p^0 \;=\; +\sqrt{\mathbf{p}^2 + M^2(\psi)} \tag{35}$$

where $M(\psi)$ is the *invariant mass* of the state $|\psi;p\rangle$. Altogether, in terms of the $|\Psi\rangle = |\psi;p\rangle$ eq. (34) becomes

$$\mathcal{F}_2(x-y) \;=\; \sum_\psi \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p},M(\psi))} \times \langle\Omega|\,\hat{\Phi}_H(x)\,|\psi,p\rangle \times \langle\psi,p|\,\hat{\Phi}_H(y)\,|\Omega\rangle\,. \tag{36}$$

Next, consider the x–dependence of the matrix element $\langle\Omega|\,\hat{\Phi}_H(x)\,|\psi,p\rangle$ and the y–dependence of the $\langle\psi,p|\,\hat{\Phi}_H(y)\,|\Omega\rangle$. The quantum field theory has translational symmetry in all 4 dimensions of spacetime, and the net energy-momentum operator $\hat{P}^\mu$ is the generator of this symmetry. In the Heisenberg picture of the theory, this means

$$\hat{\Phi}_H(x+a) \;=\; \exp(+ia_\mu\hat{P}^\mu)\,\hat{\Phi}_H(x)\,\exp(-ia_\mu\hat{P}^\mu) \tag{37}$$

and in particular

$$\hat{\Phi}_H(x) \;=\; \exp(+ix_\mu\hat{P}^\mu)\,\hat{\Phi}_H(0)\,\exp(-ix_\mu\hat{P}^\mu). \tag{38}$$

At the same time, the states $\langle\Omega|$ and $|\psi,p\rangle$ are eigenstates of the net energy-momentum operators: the vacuum $\langle\Omega|$ has $P=0$ while the state $|\psi,p\rangle$ has $P=p$. Consequently,

$$\langle\Omega|\exp(ix_\mu\hat{P}^\mu) \;=\; \langle\Omega| \quad\text{while}\quad \exp(-ix_\mu\hat{P}^\mu)\,|\psi,p\rangle \;=\; e^{-ix_\mu p^\mu}\times|\psi,p\rangle\,, \tag{39}$$

and therefore

$$\langle\Omega|\,\hat{\Phi}_H(x)\,|\psi,p\rangle \;=\; \langle\Omega|\exp(ix_\mu\hat{P}^\mu)\,\hat{\Phi}_H(0)\,\exp(-ix_\mu\hat{P}^\mu)\,|\psi,p\rangle \;=\; e^{-ix_\mu p^\mu}\times\langle\Omega|\,\hat{\Phi}_H(0)\,|\psi,p\rangle\,. \tag{40}$$

Similarly,

$$\langle\psi,p|\,\hat{\Phi}_H(y)\,|\Omega\rangle \;=\; \langle\psi,p|\exp(iy_\mu\hat{P}^\mu)\,\hat{\Phi}_H(0)\,\exp(-iy_\mu\hat{P}^\mu)\,|\Omega\rangle \;=\; e^{+iy_\mu p^\mu}\times\langle\psi,p|\,\hat{\Phi}_H(0)\,|\Omega\rangle\,. \tag{41}$$

Combining these two formulae, we have

$$\begin{aligned}\langle\Omega|\,\hat{\Phi}_H(x)\,|\psi,p\rangle\times\langle\psi,p|\,\hat{\Phi}_H(y)\,|\Omega\rangle \;&=\; e^{-ipx+ipy}\times\langle\Omega|\,\hat{\Phi}_H(0)\,|\psi,p\rangle\,\langle\psi,p|\,\hat{\Phi}_H(0)\,|\Omega\rangle\\ &=\; e^{-ip(x-y)}\times\left|\langle\psi,p|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2\end{aligned} \tag{42}$$

where only the $e^{-ip(x-y)}$ factor depends on the x and y coordinates. Moreover, it's the only factor depending on the total momentum p! Indeed, the state $\hat{\Phi}_H(0)\,|\Omega\rangle$ is invariant under orthochronous Lorentz symmetries, hence

$$\text{the matrix element }\ \langle\psi,p|\,\hat{\Phi}_H(0)\,|\Omega\rangle\ \text{ is the same for all }\mathbf{p}\in\text{the mass shell.} \tag{43}$$

Renaming this $\mathbf{p}$-independent matrix element as simply $\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle$, we have

$$\langle\Omega|\,\hat{\Phi}_H(x)\,|\psi,p\rangle\times\langle\psi,p|\,\hat{\Phi}_H(y)\,|\Omega\rangle \;=\; \left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2\times e^{-ip(x-y)}. \tag{44}$$

Consequently, eq. (36) for the two-point correlation function becomes

$$\begin{aligned}\mathcal{F}_2(x-y) \;&=\; \sum_\psi\left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2\times\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2p^0}\,e^{-ip(x-y)}\bigg|^{p^0=+\sqrt{\mathbf{p}^2+M^2(\psi)}}\\ &=\; \sum_\psi\left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2\times D(x-y;M(\psi)).\end{aligned} \tag{45}$$

Now remember that eq. (45) follows from eq. (34), which obtains only for $x^0>y^0$. In the opposite case of $x^0 < y^0$, we have

$$\begin{aligned}\mathcal{F}_2(x-y) &= \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \;=\; \langle\Omega|\,\hat{\Phi}_H(y)\hat{\Phi}_H(x)\,|\Omega\rangle \\ &= \sum_\psi\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p},M(\psi))}\,\langle\Omega|\,\hat{\Phi}_H(y)\,|\psi,p\rangle\times\langle\psi,p|\,\hat{\Phi}_H(x)\,|\Omega\rangle\,,\end{aligned} \tag{46}$$

similar to eq. (36) but with x and y exchanging their roles. Consequently, proceeding exactly as above, we obtain

$$\mathcal{F}_2(x-y) = \sum_\psi\left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2\times D(y-x;M(\psi)). \tag{47}$$

Altogether, for any time order of the x^0 and the y^0, we have

$$\begin{aligned}\mathcal{F}_2(x-y) &= \sum_\psi\left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2\times\begin{cases}D(x-y;M(\psi)) & \text{for } x^0>y^0,\\ D(y-x;M(\psi)) & \text{for } x^0<y^0,\end{cases}\\ &= \sum_\psi\left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2\times G_F(x-y;M(\psi)) \\ &= \sum_\psi\left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2\times\int\frac{d^4p}{(2\pi)^4}\,\frac{ie^{-ip(x-y)}}{p^2-M^2(\psi)+i\epsilon}\,.\end{aligned} \tag{48}$$

Eq. (48) is usually written as the Källén–Lehmann spectral representation:

$$\mathcal{F}_2(x-y) = \int\limits_0^\infty\frac{dm^2}{2\pi}\,\rho(m^2)\times\int\frac{d^4p}{(2\pi)^4}\,\frac{ie^{-ip(x-y)}}{p^2-m^2+i\epsilon}\,. \tag{49}$$

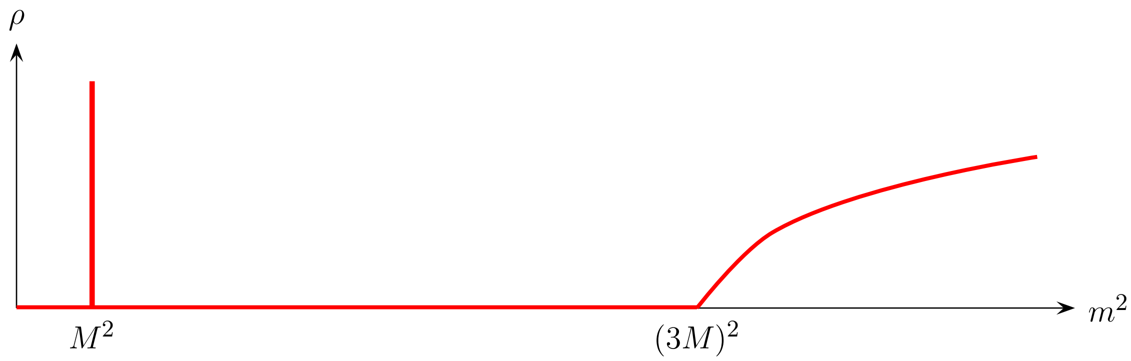
where

$$\rho(m^2) \stackrel{\text{def}}{=} \sum_\psi\left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2\times(2\pi)\delta(M^2(\psi)-m^2) \tag{50}$$

is the *spectral density function.* Here are some of its key features:

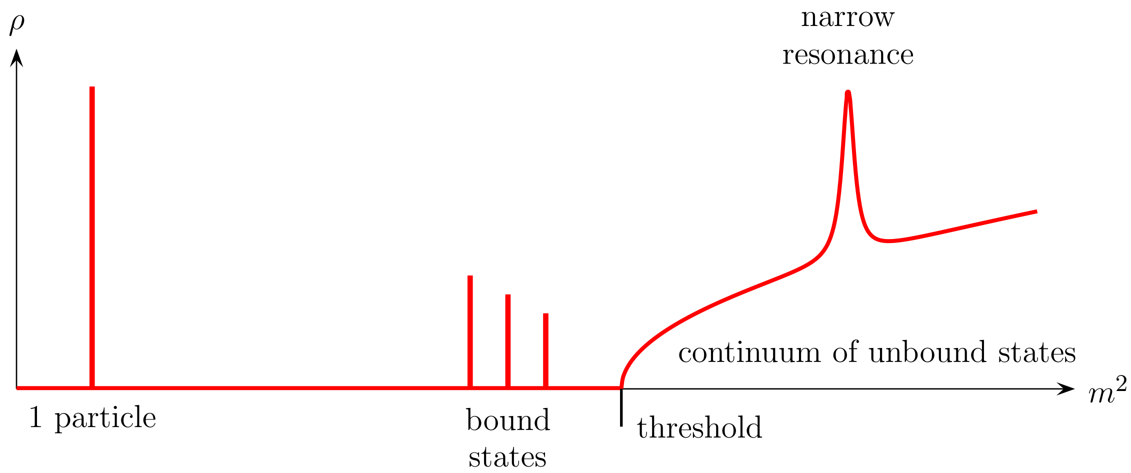
- In any QFT, for any quantum field, the spectral density function is real and non-negative, $\rho(m^2)\geq 0$ at all m^2.
- In the free field theory, $\rho(m^2)=2\pi\delta(m^2-M^2)$ where M is the particle's mass.

- In the interacting $\lambda\Phi^4$ theory, the spectral density function has both a delta-spike at $m^2 = M^2$ and a smooth continuum above the 3-particle threshold,



  ⋆ Note: the M^2 position of the delta-spike is the mass^2 of the physical particle rather than the bare mass^2 in the Feynman rules of the perturbation theory. Likewise, the continuum begins at $(3M)^2$, which is the threshold for the mass^2 for the physical 3-particle states.

- In a general quantum field theory, the spectral density functions get contributions from several kind of states: single particle, bound states of several particles, unbound states, unstable resonances, *etc.*, *etc.* The single-particle states and the bound states give rise to the delta-spikes of the spectral density function, the un-bound multi-particle states give rise to the continuum starting at the threshold (the minimal invariant mass^2 of the unbound state), while the resonances give rise to narrow peaks on top of the continuum. Schematically,



Analytic Behavior

Now let's translate all these features of the spectral density function $\rho(m^2)$ into the analytic behavior of the two-point function $\mathcal{F}_2(x-y)$, or rather of its Fourier transform

$$\mathcal{F}_2(p) \;=\; \int\! d^4x\, e^{ip_\mu x^\mu}\, \mathcal{F}_2(x-0) \;=\; \int\limits_0^\infty \frac{dm^2}{2\pi}\, \rho(m^2) \times \frac{i}{p^2 - m^2 + i\epsilon}\,. \tag{51}$$

In the $\lambda\Phi^4$ theory, the one-particle state contributes a delta-spike to the spectral density function, while the multi-particle unbound states give rise to a smooth continuum, thus

$$\rho(m^2) \;=\; Z \times 2\pi\delta(m^2 - M^2_{\rm particle}) \;+\; \text{smooth continuum}, \tag{52}$$

where

$$Z \;=\; \Big|\langle 1\,\text{particle}|\, \hat\Phi_H(0)\, |\Omega\rangle\Big|^2 \;>\; 0. \tag{53}$$

In other words, $\sqrt{Z}$ is the strength with which the quantum field $\hat\Phi$ creates single particles from the vacuum. In the free theory $Z = 1$, but in the interacting theory Z is subject to quantum corrections.

Plugging the spectral density (52) into eq. (51) for the two-point function, we get

$$\mathcal{F}_2(p^2) \;=\; \frac{iZ}{p^2 - M^2_{\rm particle} + i\epsilon} \;+\; \text{smooth}(p^2) \tag{54}$$

where the pole is at the physical particle's mass^2 and its residue Z is the square of the field's strength in creating that particle. Conversely, if we find — from the perturbation theory, or by any other means — that the two-point function has a pole at $p^2 = M^2$ with residue Z, then the spectral density function has a delta-spike just like in eq. (52), which means that the pole position M^2 is precisely the physical mass of the particle!

In perturbation theory, the Feynman vertices use the bare coupling $\lambda_{\rm bare}$ which is different from the physical coupling $\lambda_{\rm phys}$ of the theory; likewise, the Feynman propagators use the bare mass $m_{\rm bare}$ which is different from the physical mass of the particle. To relate the bare mass to the physical mass, we should use the perturbation theory to calculate the two-point correlation function $\mathcal{F}_2(p^2)$. That two-point function should have a pole, generally at $M^2_{\rm pole} \neq m^2_{\rm bare}$. It is that pole mass^2 $M^2_{\rm pole}$ which should be identified with the physical mass^2 of the particle! In other words, we should get the pole mass^2 as a perturbative expansion

$$M^2_{\rm pole} \;=\; m^2_{\rm bare} \;+\; \text{loop corrections} \;=\; f(m^2_{\rm bare}, \lambda_{\rm bare}, \Lambda_{UV}), \tag{55}$$

and then we should identify $M^2_{\rm pole} = M^2_{\rm particle}$ and solve the equation

$$f(m^2_{\rm bare}; \text{other stuff}) \;=\; M^2_{\rm particle} \tag{56}$$

for the $m^2_{\rm bare}$. We shall see how this works in practice later in these notes.

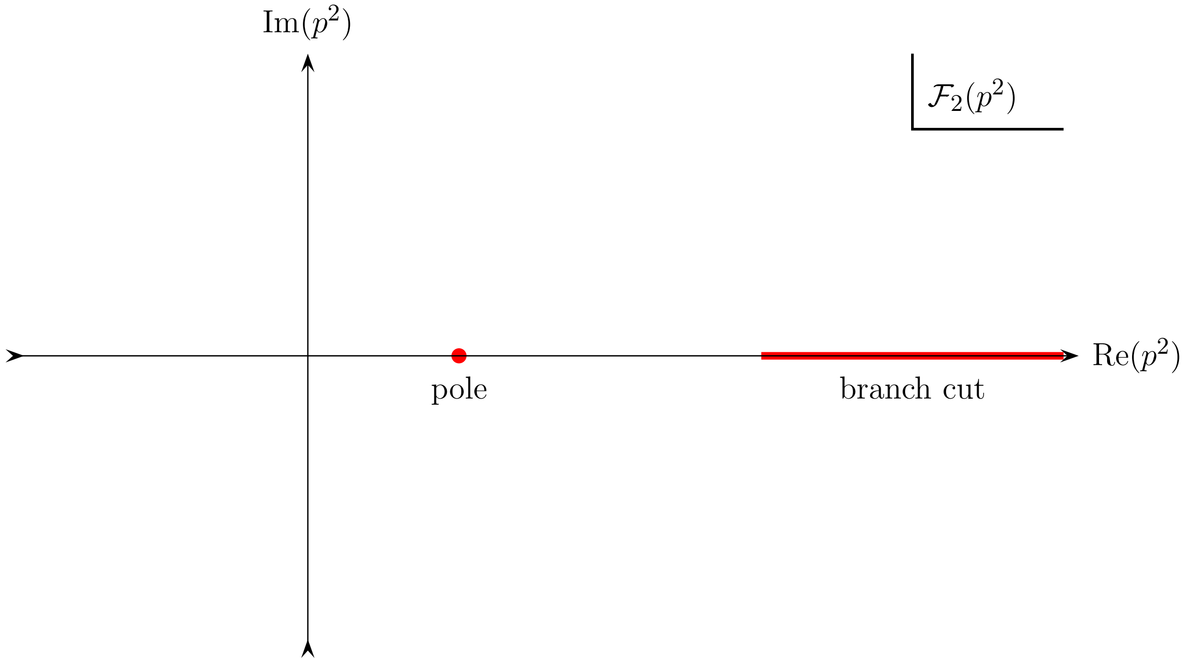
Meanwhile, let's consider the un-bound states contribution to the two-point function. In the integral

$$\mathcal{F}_2(p^2) \;=\; \int\limits_0^\infty \frac{dm^2}{2\pi}\, \rho(m^2) \times \frac{i}{p^2 - m^2 + i\epsilon}\,, \tag{51}$$

a smooth positive $\rho(m^2)$ above the threshold $m^2_{\rm thr} = 9M^2$ gives rise to the branch cut running from the threshold to $+\infty$ along the real axis. Indeed, for $p^2 >$ the threshold so that $\rho(m^2 = p^2)$ is positive and smooth, we have

$$\begin{aligned}
\mathcal{F}_2(p^2 + i\epsilon) \;-\; \mathcal{F}_2(p^2 - i\epsilon) \;&=\; \frac{1}{2\pi i}\int\limits_0^\infty \frac{dm^2\,\rho(m^2)}{m^2 - p^2 - i\epsilon} \;-\; \frac{1}{2\pi i}\int\limits_0^\infty \frac{dm^2\,\rho(m^2)}{m^2 - p^2 + i\epsilon} \\
&=\; \frac{1}{2\pi i} \oint\limits_{\substack{\text{circle}\\ \text{around } p^2}} \frac{dm^2\,\rho(m^2)}{m^2 - p^2} \;=\; \rho(p^2).
\end{aligned} \tag{57}$$

Thus, the 2-point function has a discontinuity across the real axis between $p^2 + i\epsilon$ and $p^2 - i\epsilon$, which means a branch cut on its Riemann surface:



Let's take a closer look at the real axis of this Riemann surface. For the real and negative (spacelike) p^2, the two-point function, or rather the

$$i\mathcal{F}_2(p^2) \;=\; \int_0^{+\infty} \frac{dm^2}{2\pi}\,\frac{\rho(m^2)}{m^2-p^2} \tag{58}$$

is real, positive, and single-valued. As we continue to the positive (timelike) p^2 but stay below the threshold, the $i\mathcal{F}_2(p^2)$ remains real and single-valued. But once we cross over the threshold, the integral (58) includes the singularity at $m^2 = p^2$, which gives rise to the branch cut. In this regime,

$$\text{(above the threshold)}\quad i\mathcal{F}_2(p^2 \pm i\epsilon) \;=\; \text{real} \;\pm\; i\,\frac{\rho(p^2)}{2}\,. \tag{59}$$

So what should we do for real p^2 above the threshold? The $i\epsilon$ in the denominator under the integral (51) gives the answer⋆: we should shift such real p^2 upward in the complex plane,

⋆ The $p^2 - m^2 + i\epsilon$ in the denominator of eq. (51) stems from the similar denominator in the Källén–Lehmann representation (49), which in turn comes from the Feynman propagator $G_F(x-y; M(\psi))$ for the scalar field, *cf.* eq. (48).

$p^2 \to p^2 + i\epsilon$, and evaluate the 2-point function for $p^2 + i\epsilon$. In other words, *the physical 'bank' of the branch cut is the upper bank.*

More generally, the Riemann surface of the 2-point function $\mathcal{F}_2(p^2)$ has the physical sheet and an infinite series of the un-physical sheets. The physical sheet begins on the upper bank of the branch cut and extends counterclockwise to the negative real axis and back to the positive axis. *On this physical sheet, the $\mathcal{F}_2(p^2)$ has no off-axis poles. Instead, all the poles are at real positive p^2 and correspond to physical stable particles (or bound states).*

However, the two-point function may have additional off-axis poles on the un-physical sheet of the Riemann surface beyond the branch cut. Such poles — if any — corresponds to the unstable particles or resonances. Specifically:

- First, we define the $\mathcal{F}(p^2) \stackrel{\text{def}}{=} \mathcal{F}_2(p^2 + i\epsilon)$ along the upper — physical — bank of the branch cut.
- Second, we analytically continue this function to complex p^2. For positive $\text{Im}(p^2)$ this continuation takes us to the physical sheet of the Riemann surface, while for the negative $\text{Im}(p^2)$ it takes us to the unphysical sheet below the branch cut.
- It is on this un-physical sheet that the two-point function may have an off-axis pole, or perhaps several poles. For example, suppose it has a pole at $p^2 = M^2 - iM\Gamma$. Mathematically, this means we start with $p^2 = M^2 + i\epsilon$, analytically continue from positive $\text{Im}\, p^2$ to negative $\text{Im}\, p^2$, and only then hit the pole at $\text{Im}\, p^2 = -M\Gamma$.
- Suppose Γ is small so the pole on the un-physical sheet is close to the real axis. Then for the real p^2 near M^2, the two-point function is dominated by that pole,

  $$\text{for } p^2 \approx M^2, \quad \mathcal{F}(p^2) \;=\; \frac{iZ}{p^2 - M^2 + iM\Gamma} \;+\; \text{smooth}(p^2). \tag{60}$$

  This is the *Breit–Wigner resonance*.
- Physically, such a resonance corresponds to an un-stable particle. By the optical theorem, the resonance's width Γ equals to the net decay rate of the unstable particle, including all possible decay products. In other words, $1/\Gamma$ is the average lifetime of the unstable particle.

Perturbation Theory for the Two–Point Function

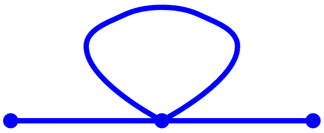
By now we have learned the expected analytical behavior of the two-point function $\mathcal{F}_2(x-y)$ — or rather of its Fourier transform $\mathcal{F}_2(p)$, — let's calculate this function in the perturbation theory. At the tree level we have a single Feynman diagram

$$\mathcal{F}_2^{\text{tree}} =$$

$$= \frac{i}{p^2 - m_b^2 + i\epsilon}, \tag{61}$$

at the one-loop level we also have a single diagram

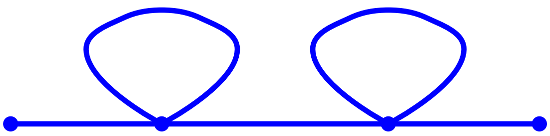
$$\mathcal{F}_2^{1\text{ loop}} =$$



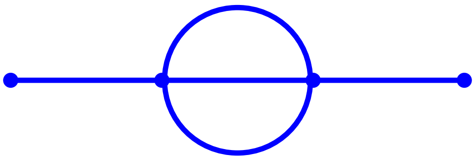
$$= \left(\frac{i}{p^2 - m_b^2 + i\epsilon}\right)^2 \times \text{other factors}, \tag{62}$$

at the two-loop level we have three diagrams,

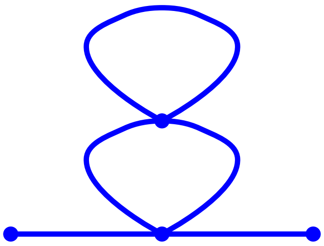
$$\mathcal{F}_2^{2\text{ loops}} =$$



$$+$$



$$+$$



$$\begin{aligned} &= \left(\frac{i}{p^2 - m_b^2 + i\epsilon}\right)^3 \times \text{other factors} \\ &\quad + \left(\frac{i}{p^2 - m_b^2 + i\epsilon}\right)^2 \times \text{other factors} + \left(\frac{i}{p^2 - m_b^2 + i\epsilon}\right)^2 \times \text{other factors}, \end{aligned} \tag{63}$$

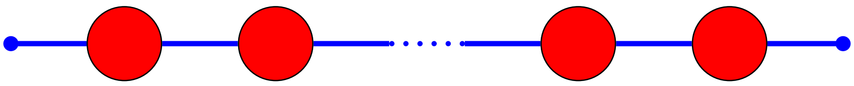
etc., *etc.* Similar to the one-loop and two-loop diagrams shown here, all higher-loop diagrams also have two or more propagators whose momentum is fixed at p, so the whole diagram depends on p as

$$\left(\frac{i}{p^2-m_b^2+i\epsilon}\right)^{\text{power}\geq 2}\times\text{other factors.}\tag{64}$$

Thus, instead shifting the tree-level pole at p^2 = bare mass^2 to the physical mass^2, the loop diagrams all have poles at the same p^2 = bare mass^2 as the tree graph. Worse, the loop diagrams have double poles, triple poles, or worse, while the physical correlation function may have simple poles but no higher-order poles. To resolve this conundrum, we need to partially re-sum the perturbative expansion so that the higher-order poles would add up to a simple but shifted pole such as in the series

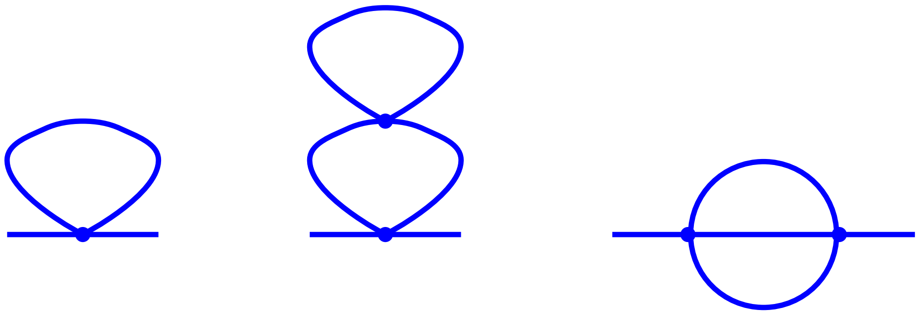
$$\sum_{n=0}^{\infty}\left(\frac{i}{p^2-m_b^2+i\epsilon}\right)^{n+1}\times(-i\Delta)^n \;=\; \frac{i}{p^2-m_b^2+i\epsilon}\times\frac{1}{1-\frac{\Delta}{p^2-m_b^2+i\epsilon}} \;=\; \frac{i}{p^2-(m_b^2+\Delta)+i\epsilon}\,.\tag{65}$$

To see how such resummation might work, note that a general multi-loop Feynman graph for the two-point correlation function looks like



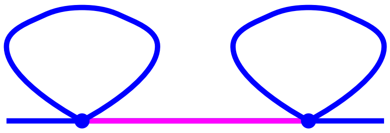
(66)

where the red circles stand for *one-particle irreducible* (1PI) subgraphs. That is, subgraphs which would remain connected after cutting any single internal propagator, hence the name one-propagator irreducible, which turned into one-particle irreducible. For example, the subgraphs



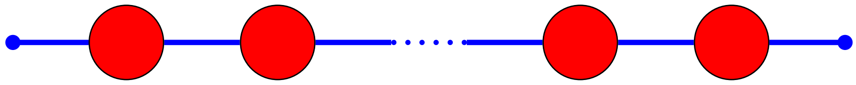
(67)

are 1PI but the subgraph



(68)

is not 1PI — cutting the purple propagator breaks the graph into 2 disconnected pieces. In a 1PI Feynman graph, none of the internal propagator's momenta is fixed by the external legs' momenta; consequently, there are no poles when the external momenta — or any combinations of the external momenta — go to the mass shell $p^2 = M^2$. Thus, when we evaluate a Feynman graph



(69)

with N 1PI bubbles connected to each other and to the external vertices by $N+1$ propagators, we get

$$\mathcal{F} \;=\; \left(\frac{i}{p^2 - m_b^2 + i\epsilon}\right)^{N+1} \times \prod_{i=1}^{N} [\text{1PI bubble}\#i] \tag{70}$$

where the bubble factors are analytic functions of p^2 which are regular at $p^2 = M^2$.

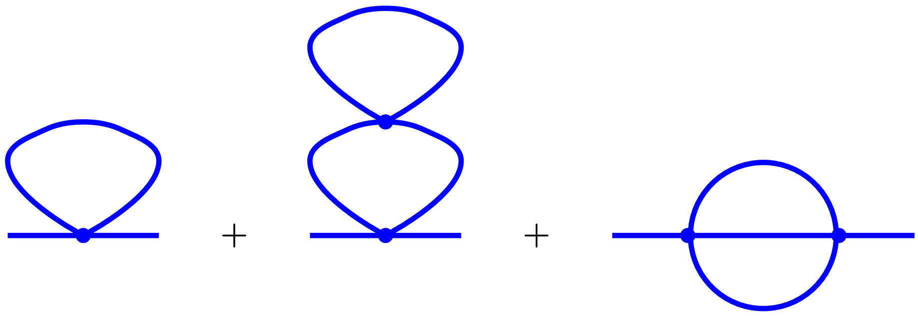
Now let's reorganize the perturbative expansion of the two-point correlation function into diagrams like (69) according to the number N of the 1PI bubbles. That is, we first formally sum up the diagrams with a fixed number N of the 1PI bubbles but allow any kinds of such bubbles, and only then sum over N. This gives us

$$\mathcal{F}_{N\,\text{bubbles}}(p^2) \;=\; \left(\frac{i}{p^2 - m_b^2 + i\epsilon}\right)^{N+1} \times \sum_{\substack{\text{sets of } N\\ \text{1PI bubbles}}} \left(\prod_{i=1}^{N} [\text{1PI bubble}\#i]\right) \tag{71}$$

and since we can choose any 1PI subgraph for each bubble$\#i$ independently from the other bubbles, the sum here factorizes to N^{th} power of the sum over single 1PI bubbles,

$$\mathcal{F}_{N\,\text{bubbles}}(p^2) \;=\; \left(\frac{i}{p^2 - m_b^2 + i\epsilon}\right)^{N+1} \times \left[\sum_{\substack{\text{single}\\ \text{1PI bubbles}}} [\text{bubble}]\right]^N . \tag{72}$$

To simplify our notations, let's denote the sum over single 1PI bubbles

$$-i\Sigma(p^2) \;=$$  (73)

$$+\ \text{higher-loop 1PI graphs,}$$

then the sum over all the N-bubble diagrams for the two-point function is

$$\mathcal{F}_{N\,\text{bubbles}}(p^2) \;=\; \left(\frac{i}{p^2-m_b^2+i\epsilon}\right)^{N+1}\times\Bigl(-i\Sigma(p^2)\Bigr)^N. \tag{74}$$

Finally, we sum over N, — and that's when the higher-order poles in the N-bubble diagrams — add up to a shifted pole in the complete two-point function

$$\begin{aligned}\mathcal{F}_2(p^2) \;&=\; \sum_{N=0}^{\infty} i\mathcal{F}_{N\,\text{bubbles}}(p^2) \;=\; \sum_{N=0}^{\infty}\left(\frac{i}{p^2-m_b^2+i\epsilon}\right)^{N+1}\times\Bigl(-i\Sigma(p^2)\Bigr)^N \\ &=\; \frac{i}{p^2-m_b^2+i\epsilon}\times\left[\sum_{N=0}^{\infty}\left(\frac{\Sigma(p^2)}{p^2-m_b^2+i\epsilon}\right)^N \;=\; \frac{1}{1\,-\,\frac{\Sigma(p^2)}{p^2-m_b^2+i\epsilon}}\right] \\ &=\; \frac{i}{p^2-m_b^2-\Sigma(p^2)+i\epsilon}\,.\end{aligned} \tag{75}$$

The specific location of the shifted pole is wherever the denominator of the above formula happens to vanish, *i.e.* the solution of the equation

$$p^2\;-\;m_b^2\;-\;\Sigma(p^2)\;=\;0. \tag{76}$$

It is this solution which gives us the physical mass^2 of the particle M^2. In other words, given the physical mass M of the particle, we should set the bare mass^2 of the perturbation theory not to M^2 but to

$$m_b^2 \;=\; M^2 \;-\; \Sigma(p^2 = M^2). \tag{77}$$

Of course, the $\Sigma(p^2)$ here itself obtains from the perturbation theory, so it implicitly depends on the bare coupling λ_b, on the bare mass^2 m_b^2, and on the UV cutoff Λ, thus $\Sigma(p^2;\lambda_b,m_b^2,\Lambda^2)$. Consequently, eq. (77) becomes a non-trivial equation

$$m_b^2 \;=\; M^2 \;-\; \Sigma(p^2 = M^2;\lambda_b,m_b^2,\Lambda^2) \tag{78}$$

which we need to solve for the bare mass parameter of the perturbation theory in order to get physically correct results.

Now consider the field strength factor Z which obtains as the residue of the pole of $\mathcal{F}_2(p^2)$ at the physical mass^2 of the particle. Let $p^2 = M^2 + \delta p^2$ for an infinitesimally small δp^2; then

$$\Sigma(p^2) \;=\; \Sigma(M^2) \;+\; \delta p^2 \times \left.\frac{d\Sigma}{dp^2}\right|_{p^2=M^2} \;+\; \tfrac{1}{2}(\delta p^2)^2 \times \left.\frac{d^2\Sigma}{(dp^2)^2}\right|_{p^2=M^2} \;+\; \cdots, \tag{79}$$

hence in the denominator of

$$\mathcal{F}_2(p^2) \;=\; \frac{i}{p^2 - m_b^2 - \Sigma(p^2) + i\epsilon} \tag{80}$$

we have

$$\begin{aligned}
p^2 \;-\; m_b^2 \;-\; \Sigma(p^2) \;&=\; M^2 \;+\; \delta p^2 \;-\; m_b^2 \;-\; \Sigma(M^2) \;-\; \delta p^2 \times \left.\frac{d\Sigma}{dp^2}\right|_{p^2=M^2} \;+\; O\big((\delta p^2)^2\big) \\
&=\; \Big(M^2 \;-\; m_b^2 \;-\; \Sigma(M^2)\Big) \;+\; \delta p^2 \times \left(1 \;-\; \left.\frac{d\Sigma}{dp^2}\right|_{p^2=M^2}\right) \;+\; O\big((\delta p^2)^2\big) \\
&\qquad \langle\!\langle \text{ where the first term} = 0 \,\rangle\!\rangle \\
&=\; \delta p^2 \times \left(1 \;-\; \left.\frac{d\Sigma}{dp^2}\right|_{p^2=M^2} \;+\; O(\delta p^2)\right)
\end{aligned} \tag{81}$$

Taking the inverse of this expression, we get

$$\begin{aligned}\mathcal{F}_2(p^2=M^2+\delta p^2) \;&=\; \frac{i}{\delta p^2}\times\left(1\,-\,\frac{d\Sigma}{dp^2}\bigg|_{p^2=M^2}\,+\,O(\delta p^2)\right)^{-1}\\ &=\; \frac{i}{\delta p^2}\times\left(\frac{1}{1\,-\,(d\Sigma/dp^2)}\,+\,O(\delta p^2)\right)\\ &=\; \frac{i}{\delta p^2}\times\frac{1}{1\,-\,(d\Sigma/dp^2)}\,+\,\text{finite.}\end{aligned} \tag{82}$$

Or in other words,

$$\mathcal{F}_2(p^2\,\text{near}\,M^2) \;=\; \frac{iZ}{p^2-M^2+i\epsilon}\,+\,\text{finite} \tag{83}$$

for

$$Z \;=\; \frac{1}{1\,-\,\frac{d\Sigma}{dp^2}\Big|_{p^2=M^2}}\,. \tag{84}$$

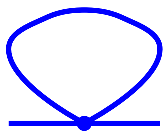
One Loop Example

Formally, $-i\Sigma(p^2)$ is the sum of all 1PI graphs with 2 external legs. In practice, we can only calculate the graphs up to some maximal number L of loops to get

$$\Sigma(p^2) \;=\; \lambda\times f_1(p^2)\;+\;\lambda^2\times f_2(p^2)\;+\;\cdots\;+\;\lambda^L\times f_L(p^2)\;+\;\Big(O(\lambda^{L+1})\ \text{unknown}\Big). \tag{85}$$

Consequently, the bare mass m_b has to be adjusted order-by-order in perturbation theory, just like we adjust the bare coupling λ_b.

So let's start with the one-loop order, in which we have only one 1PI graph to calculate, namely

$$-i\Sigma_{1\,\text{loop}}(p^2) \;=$$  (86)

Evaluating this graph, we get

$$-i\Sigma_{1\,\text{loop}}(p^2) \;=\; \frac{-i\lambda}{2}\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m_b^2+i\epsilon}\,, \tag{87}$$

and before we calculate the integral on the RHS, we immediately see that it does not depend on the momentum p. Consequently, at the one-loop order we get the mass shift according to

$$m_b^2 \;=\; M^2 \;-\; \Sigma \tag{88}$$

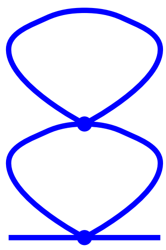
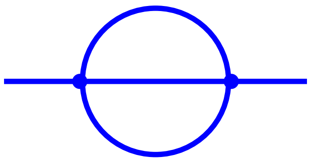
but no field strength renormalization,

$$\frac{d\Sigma_{1\,\text{loop}}}{dp^2} \;=\; 0 \quad\Longrightarrow\quad Z_{1\,\text{loop}} \;=\; 1. \tag{89}$$

This behavior is peculiar to the $\lambda\phi^4$ theory — in other theories, quantum corrections to Z begin at the one-loop order, as we shall see next class for the Yukawa theory. But in the $\lambda\phi^4$ theory, corrections to the field strength Z happen to start at the two loop order,

$$Z \;=\; 1 \;+\; O(\lambda^2). \tag{90}$$

Indeed, there are two 2-loop 1PI graphs for the $\Sigma(p^2)$, namely

$-i\Sigma_{2\,\text{loop}}(p^2) \;=$  $+$  (91)

and in the second graph here — and only in the second graph — the internal momenta do depend on p, so it yields a non-zero contribution to $d\Sigma/dp^2$ and hence to $Z - 1$. I leave the calculation of that graph for your homework set#15, so get ready to work hard.

For the moment, let's focus on the one-loop graph (86) and its momentum integral

$$\int \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2 - m_b^2 + i\epsilon}\,.$$

Wick-rotating the loop momentum q to the Euclidean momentum space, we get

$$d^4q \;\to\; i d^4q_E\,; \quad \frac{i}{q^2 - m_b^2 + i\epsilon} \;=\; \frac{i}{-q_E^2 - m_b^2} \;=\; \frac{-i}{q_E^2 + m_b^2} \tag{92}$$

and hence

$$\Sigma_{1\,\text{loop}} \;=\; \frac{\lambda}{2}\int\frac{d^4q_E}{(2\pi)^4}\,\frac{1}{q_E^2+m_b^2}\,. \tag{93}$$

The integral here diverges quadratically at $q_E \to \infty$, so let's regulate it using Wilson's hard-edge cutoff instead of dimensional regularization. Thus,

$$\begin{aligned}
\int\limits_{\text{reg}}\frac{d^4q_E}{(2\pi)^4}\,\frac{1}{q_E^2+m_b^2} \;&=\; \int\limits_0^\Lambda dq_E\times q_E^3\times\frac{1}{q_E^2+m_b^2}\times\int\frac{d^3\Omega(q_E^\mu)}{(2\pi)^4}\\
&=\; \frac{2\pi^2}{(2\pi)^4}\times\int\limits_0^{\Lambda^2}\tfrac{1}{2}dq_E^2\,\frac{q_E^2}{q_E^2+m_b^2}\\
&=\; \frac{1}{16\pi^2}\int\limits_{m_b^2}^{\Lambda^2+m_b^2}dt\,\frac{t-m_b^2}{t}\\
&=\; \frac{1}{16\pi^2}\left(\Lambda^2\;-\;m_b^2\times\log\frac{\Lambda^2+m_b^2}{m_b^2}\right)\\
&=\; \frac{1}{16\pi^2}\left(\Lambda^2\;-\;m_b^2\times\log\frac{\Lambda^2}{m_b^2}\;+\;O(m_b^4/\Lambda^2)\right).
\end{aligned} \tag{94}$$

and hence

$$\Sigma_{1\,\text{loop}} \;=\; \frac{\lambda}{32\pi^2}\left(\Lambda^2\;-\;m_b^2\times\log\frac{\Lambda^2}{m_b^2}\right). \tag{95}$$

In terms of the difference between the bare and the physical masses, this means

$$M_{\text{phys}}^2 \;=\; m_b^2\;+\;\frac{\lambda}{32\pi^2}\left(\Lambda^2\;-\;m_b^2\times\log\frac{\Lambda^2}{m_b^2}\right)\;+\;O(\lambda^2). \tag{96}$$

The $(\lambda/32\pi^2)\Lambda^2$ on the RHS here stems from the quadratic UV divergence of the momentum integral, and it causes the *fine tuning* problem of loop corrections to the bare mass being much larger than the mass itself. Indeed, we always take the UV cutoff scale to be much larger than the particle's mass, $\Lambda \gg M$, so even for a rather small coupling λ we may end up with

$$M^2\;-\;m_b^2\;\sim\;\frac{\lambda\Lambda^2}{32\pi^2}\;\gg\;M^2. \tag{97}$$

For example, consider the Higgs particle of the Standard model: it has mass $M \approx 125$ GeV and self-coupling $\lambda \approx 0.25$. So if we set the UV cutoff just above the LHC reach at $\Lambda = 10$ TeV, we would end up with the one-loop correction to the Higgs field's bare mass $\Delta m^2 \approx (280\text{ GeV})^2$, larger than the mass itself. And that was just the correction due to Higgs field interacting with itself; the corrections due to its interactions with the other fields — especially the $W^\pm$ and Z^0 vector fields and the top quark — are even stronger than the corrections due to self interaction.

These large corrections mean that the physical mass^2 of the particle comes out as a small difference between a large bare mass^2 and a large quantum correction. So if we want to end up with the physical mass in the same ballpark as the experimental data, we would need to *fine tune* the m_b^2 parameter of the perturbation theory to a high precision. And the higher we set the UV cutoff scale Λ, the worse the fine tuning problem becomes. For example, if we set Λ to the Grand Unification scale of 10^{16} GeV and λ_b to 0.3, then the quantum correction to the particle mass would be about

$$M^2 \;-\; m_b^2 \;\sim\; 10^{29}\ \text{GeV}^2. \tag{98}$$

Hence, if we want the physical mass M to be in the 100 GeV ballpark, we would need to fine tune the m_b^2 parameter to the accuracy of 25 significant figures! Moreover, at the higher loop orders we would also get large quantum corrections of the order $(\lambda/32\pi^2)^{\#\text{loops}} \times \Lambda^2$, so we would need to adjust the already fine-tuned m_b^2 parameter order by order in perturbations theory, up to about ten-loop order!

Aside: Regulator Dependence of Quadratic Divergences

Regularization of the quadratic UV divergences is much more sensitive to the details of the cutoff that the logarithmic divergences. Indeed, consider the one-loop mass shift

$$\Sigma \;=\; \frac{\lambda}{32\pi^2}\left(\Lambda_{HE}^2 \;-\; m_b^2\times\log\frac{\Lambda_{HE}^2}{m_b^2}\right) \tag{99}$$

we have calculated using the hard-edge cutoff. Now consider some other UV cutoff whose scale Λ is equivalent to the hard-edge cutoff scale as $\Lambda_{HE}^2 = C\times\Lambda^2$ for some $O(1)$ constant C. If we simply plug this relation into eq. (99), we would get

$$\Sigma \;=\; \frac{\lambda}{32\pi^2}\left(C\times\Lambda^2 \;-\; m_b^2\times\log\frac{\Lambda^2}{m_b^2} \;-\; m_b^2\times\log(C)\right). \tag{100}$$

But actually, using another cutoff would lead to

$$\Sigma \;=\; \frac{\lambda}{32\pi^2}\left(a\times\Lambda^2 \;-\; m_b^2\times\log\frac{\Lambda^2}{m_b^2} \;-\; b\times m_b^2\right) \tag{101}$$

for some $O(1)$ constants a and b, which are generally not related to the ratio C of the equivalent cutoff scales. Thus, the coefficient of the leading Λ^2 divergence is not universal, so the $O(\Lambda^2)$ correction to the bare mass^2 of the theory depends on the type of the UV cutoff! On the other hand, the coefficient of the subleading $\log\Lambda^2$ divergence is universal — it is the same for all regularization schemes.

In particular, we shall see in a moment that in dimensional regularization $a = 0$, so we do not get the leading quadratic divergence at all, only the subleading logarithmic divergence. However, the dimensional regularization has its own way to signal that the divergence of some amplitude is worse than logarithmic. Specifically, the dimension-dependent amplitude gets a pole at some $D < 4$ in additional to the usual pole at $D = 4$.

To see how this works, consider the dimensional regularization of the mass shift,

$$\Sigma(D) \;=\; \frac{\lambda}{2}\times\int\frac{\mu^{4-d}\,d^Dq_E}{(2\pi)^D}\,\frac{1}{q_E^2+m_b^2}\,. \tag{102}$$

The momentum integral here diverges for any $D \geq 2$, so we need to evaluate it for a low dimension $D < 2$ and then analytically continue the result all the way back to $D = 4$.

To evaluate the momentum integral in a non-integer dimension, we need to relate it to a Gaussian integral, so let's use

$$\frac{1}{q_E^2+m_b^2} \;=\; \int\limits_0^\infty dt\,e^{-t(q_E^2+m_b^2)}. \tag{103}$$

Consequently,

$$\begin{aligned}\Sigma(D) &= \frac{\lambda}{2}\,\mu^{4-D}\int\frac{d^Dq_E}{(2\pi)^D}\int\limits_0^\infty dt\,e^{-t(q_E^2+m_b^2)}\\ &= \frac{\lambda}{2}\,\mu^{4-D}\int\limits_0^\infty dt\,e^{-tm_b^2}\times\int\frac{d^Dq_E}{(2\pi)^D}\,e^{-tq_E^2}\\ &= \frac{\lambda}{2}\,\mu^{4-D}\int\limits_0^\infty dt\,e^{-tm_b^2}\times(4\pi t)^{-D/2}\,.\end{aligned}\tag{104}$$

In the last integral over t here, the integrand behaves as $t^{-D/2}$ for small t, so the integral diverges for $D \geq 2$. This is the t-integral's way to indicate the UV divergence: in terms of the $\int dt$ it translates to a divergence at $t \to 0$. So we need to analytically continue the mass shift Σ to $D < 2$ dimensions, and then the t integral evaluates to

$$\Sigma(D) = \frac{\lambda\mu^{4-D}}{2(4\pi)^{D/2}}\times\Gamma\big(1-\tfrac{D}{2}\big)(m_b^2)^{(D/2)-1}\tag{105}$$

Given this analytic formula, we may continue it to $D \geq 2$ and even to complex D, so let's look at the poles of $\Sigma(D)$ in the complex D plane. The $\Gamma(x)$ function has poles at $x = 0, -1, -2, -3, \ldots$, so $\Sigma(D)$ has poles at $D = 2$, $D = 4$, $D = 6$, *etc.* The pole at $D = 4$ and the poles at higher dimensions are common for all the UV-divergent amplitudes. But the pole at $D = 2$ — or for other amplitudes, any pole at $D < 4$ — indicates that the UV divergence is worse than logarithmic.

Now let's analytically continue the mass shift (105) to $D = 4$, or rather to $D = 4 - 2\epsilon$:

$$\Sigma = \frac{\lambda}{32\pi^2}\,(4\pi\mu^2)^\epsilon(m_b^2)^{1-\epsilon}\Gamma(\epsilon-1) = \frac{\lambda m_b^2}{32\pi^2}\left(\frac{4\pi\mu^2}{m_b^2}\right)^\epsilon\Gamma(\epsilon-1).\tag{106}$$

For small ϵ we have

$$\Gamma(\epsilon-1) = \frac{\Gamma(\epsilon)}{\epsilon-1} = \frac{-1}{1-\epsilon}\times\left(\frac{1}{\epsilon}-\gamma_E+O(\epsilon)\right) = -\left(\frac{1}{\epsilon}-\gamma_E+1+O(\epsilon)\right),$$

while

$$\left(\frac{4\pi\mu^2}{m_b^2}\right)^\epsilon = \exp\left(\epsilon\times\log\frac{4\pi\mu^2}{m_b^2}\right) = 1+\epsilon\times\log\frac{4\pi\mu^2}{m_b^2}+O(\epsilon^2),\tag{107}$$

hence

$$\Gamma(\epsilon-1)\times\left(\frac{4\pi\mu^2}{m_b^2}\right)^{\epsilon} \;=\; -\left(\frac{1}{\epsilon}\,-\,\gamma_E\,+\,1\,+\,\log\frac{4\pi\mu^2}{m_b^2}\,+\,O(\epsilon)\right), \tag{108}$$

where the $O(\epsilon)$ term can be neglected in the $\epsilon\to 0$ limit.

Altogether, we end up with

$$\Sigma \;=\; -\frac{\lambda m_b^2}{32\pi^2}\times\left(\frac{1}{\epsilon}\,-\,\gamma_E\,+\,1\,+\,\log\frac{4\pi\mu^2}{m_b^2}\right), \tag{109}$$

which we may interpret as

$$\Sigma \;=\; -\frac{\lambda m_b^2}{32\pi^2}\times\left(\log\frac{\Lambda_{\text{eff}}^2}{m_b^2}\;+\;\text{const}\right). \tag{110}$$

As promised, the dimensional regularization yields the correct subleading logarithmic divergence but misses the leading quadratic divergence. Although the pole at $D=2$ does give a warning sign that the divergence is worse than logarithmic.

Yukawa Theory as Example of $Z\neq 1$

As an example of a theory where we can see the field strength renormalization at one-loop order, let's consider the Yukawa theory: A Dirac fermion field Ψ and a real scalar field Φ governed by the bare Lagrangian

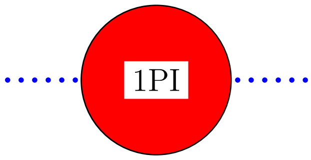
$$\mathcal{L} \;=\; \overline{\Psi}(i\,\partial\!\!\!/-m_f)\Psi\;+\;\tfrac{1}{2}(\partial_\mu\Phi)^2\;-\;\tfrac{1}{2}m_s^2\Phi^2\;+\;g\Phi\overline{\Psi}\Psi, \tag{111}$$

where the last term $g\Phi\overline{\Psi}\Psi$ is treated as a perturbation. Since the Yukawa theory has two unrelated fields, it also has separate two-point correlation functions $\mathcal{F}_\Phi(p^2)$ and $\mathcal{F}_\Psi(p\!\!\!/)$, hence separate physical masses M_f and M_b and separate field strength factors Z_Φ and Z_Ψ. In these notes I focus on the scalar field's correlation function

$$\mathcal{F}_\Phi(p^2) \;=\; \frac{i}{p^2 - m_s^2 - \Sigma_\Phi(p^2) + i\epsilon} \tag{112}$$

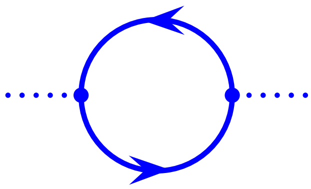
for

$$-i\Sigma_\Phi(p^2) \;=\; \tag{113}$$



At the one-loop level, there is only one 1PI graph contributing to the Σ_Φ, thus

$$-i\Sigma_\Phi^{\text{1 loop}} \;=\; \tag{114}$$



which evaluates to

$$-i\Sigma_\Phi^{\text{1 loop}}(p^2) \;=\; -\int \frac{d^4q_1}{(2\pi)^4}\,\text{Tr}\left(\frac{i}{\not q_1 - m_f + i\epsilon}\,(-ig)\,\frac{i}{\not q_2 - m_f + i\epsilon}\,(-ig)\right) \tag{115}$$

for $q_2 = p + q_1$. To calculate this integral, we start with the trace

$$\begin{aligned} \text{Trace} \;&=\; \text{Tr}\left(\frac{\not q_1 + m_f}{q_1^2 - m_f^2 + i\epsilon}\,g\,\frac{\not q_2 + m_f}{q_2^2 - m_f^2 + i\epsilon}\,g\right) \;=\; \frac{g^2\,\text{tr}\big[(\not q_1 + m_f)(\not q_2 + m_f)\big]}{(q_1^2 - m_f^2 + i\epsilon)(q_2^2 - m_f^2 + i\epsilon)} \\ &=\; g^2\,\frac{4(q_1q_2) + 4m_f^2}{(q_1^2 - m_f^2 + i\epsilon)(q_2^2 - m_f^2 + i\epsilon)}\,. \end{aligned} \tag{116}$$

Next, we use Feynman parameter trick to bring the denominator here to the form

$$\frac{1}{\text{denominator}} \;=\; \int_0^1 \frac{d\xi}{\mathcal{D}^2} \tag{117}$$

for

$$\begin{aligned} \mathcal{D} \;&=\; (1-\xi)\times(q_1^2 - m_f^2 + i\epsilon) \;+\; \xi\times(q_2^2 - m_f^2 + i\epsilon) \\ &=\; (1-\xi)q_1^2 \;+\; \xi(q_2 = q_1 + p)^2 \;-\; m_f^2 + i\epsilon \\ &=\; q_1^2 \;+\; 2\xi(q_1p) \;+\; \xi p^2 \;-\; m_f^2 \;+\; i\epsilon \\ &=\; (q_1 + \xi p)^2 \;+\; \xi(1-\xi)p^2 \;-\; m_f^2 \;+\; i\epsilon, \end{aligned} \tag{118}$$

which we may rewrite as

$$\mathcal{D} \;=\; k^2 \;-\; \Delta(\xi) \;+\; i\epsilon \tag{119}$$

for

$$k \;=\; q_1 \;+\; \xi p \quad \text{and} \quad \Delta(\xi) \;=\; m_f^2 \;-\; \xi(1-\xi)p^2. \tag{120}$$

In terms of the shifted momentum variable k, the propagator momenta are

$$q_1 \;=\; k \;-\; \xi p, \quad q_2 \;=\; k \;+\; (1-\xi)p, \tag{121}$$

hence the numerator of the trace (116) becomes

$$\begin{aligned} 4(q_1 q_2) \;+\; 4m_f^2 \;&=\; 4(k-\xi p)_\mu (k+(1-\xi)p)^\mu \;+\; 4m_f^2 \\ &=\; 4k^2 \;+\; 4(1-2\xi)(kp) \;-\; 4\xi(1-\xi)p^2 \;+\; 4m_f^2 \\ &=\; 4k^2 \;+\; 4\Delta(\xi) \;+4(1-2\xi)(kp). \end{aligned} \tag{122}$$

Collecting all these formulae and plugging them into eq. (115), we arrive at

$$\begin{aligned} \Sigma_\Phi(p^2) \;&=\; -4ig^2 \int\limits_{\text{reg}} \frac{d^4 q_1}{(2\pi)^4} \int\limits_0^1 d\xi\, \frac{k^2 + \Delta + (1-2\xi)(kp)}{[k^2 - \Delta + i\epsilon]^2} \\ &=\; -4ig^2 \int\limits_0^1 d\xi \int\limits_{\text{reg}} \frac{d^4 k}{(2\pi)^4} \frac{k^2 + \Delta + (1-2\xi)(kp)}{[k^2 - \Delta + i\epsilon]^2}\,, \end{aligned} \tag{123}$$

where on the second line we have integrated over the loop momentum before integrating over ξ and then shifted the momentum integration variable from q_1 to $k = q_1 + \xi p$. Note: for large k, the integrand here behaves as $1/k^2$, so the integral suffers from a quadratic UV divergence, which must be regulated (cut off). Thus, for any UV cutoff we could use here, we must be careful restating it in terms of the shifted momentum k rather than the propagator momenta q_1 and q_2. For the logarithmically divergent integrals such details of the cutoff can be neglected, but for the worse-than-logarithmic divergences they become important. For example, a quadratic $O(\Lambda^2)$ divergence may turn into

$$O(\Lambda^2) \;+\; O(\Lambda p) \;+\; O(p^2). \tag{124}$$

where the subleading terms depends on the details of the cutoff in terms of the shifted momentum k.

Fortunately, in dimensional regularization this problem does not arise since the DR cutoff does not affect the integrand or the range of k^μ — it stays infinite, both before and after shifting the momentum variable. Only the measure of the d^4k integral changes to $\mu^{4-D}d^Dk$, but that is not affected by shifting the momentum variable from q_1 to k. And that's why we are going to use the dimensional regularization to calculate the Σ_Φ in these notes.

But first, let's simplify the integral (123) a bit using the $k^\mu \to -k^\mu$ symmetry of the integration range — which in DR is the whole momentum space. The denominator of the integral also has this symmetry since it depends on the k^μ only via k^2. As to the numerator, the k^2 and the Δ terms are invariant under this symmetry, but the $(1-2\xi)(kp)$ term flips sign. Consequently, that term by itself would integrate to zero, so we may just as well truncate the numerator to the symmetric terms $k^2 + \Delta$, thus

$$\Sigma_\Phi \;=\; -4ig^2 \int\limits_0^1 d\xi \int \frac{d^4k}{(2\pi)^4}\, \frac{k^2+\Delta}{[k^2-\Delta+i\epsilon]^2}\,. \tag{125}$$

At this point, k^μ is the Minkowski momentum, but we are ready to Wick-rotate it to the Euclidean momentum space:

$$d^4k \;\to\; id^4k_E\,, \quad (k^2+\Delta) \;\to\; (\Delta-k_E^2), \quad (k^2-\Delta+i\epsilon) \;\to\; -(k_E^2+\Delta), \tag{126}$$

and therefore

$$\Sigma_\Phi \;=\; 4g^2 \int\limits_0^1 d\xi \int \frac{d^4k_E}{(2\pi)^4}\, \frac{\Delta-k_E^2}{(k_E^2+\Delta)^2}\,. \tag{127}$$

In 4 Euclidean dimensions this integral diverges quadratically since the integrand behaves as $1/k_E^2$ at large k_E. But but before we reduce the dimension to evaluate this integral, let's see how the UV divergence affects the p-dependence of the $\Sigma_\Phi(p^2)$. That is, let's take the derivatives

$$\frac{d\Sigma_\Phi}{dp^2}\,,\quad \frac{d^2\Sigma_\Phi}{(dp^2)^2}\,,\quad \ldots, \tag{128}$$

and check what kinds of UV divergences do they suffer from — or if they diverge at all.

Note that the integral (127) depends on the scalar's momentum p only via $\Delta(p) = m_f^2 - \xi(1-\xi)p^2$. Consequently, in the context of $\Sigma(p^2)$ and its derivatives,

$$\frac{\partial(\text{integrand})}{\partial p^2} = -\xi(1-\xi)\frac{\partial(\text{integrand})}{\partial\Delta}\,. \tag{129}$$

Thus,

$$\frac{d\Sigma_\Phi}{dp^2} = -4g^2\int_0^1 d\xi\,\xi(1-\xi)\times\int\frac{d^4k_E}{(2\pi)^4}\,\frac{\partial}{\partial\Delta}\left(\frac{\Delta-k_E^2}{(k_E^2+\Delta)^2}\right), \tag{130}$$

$$\frac{d^2\Sigma_\Phi}{(dp^2)^2} = +4g^2\int_0^1 d\xi\,\xi^2(1-\xi)^2\times\int\frac{d^4k_E}{(2\pi)^4}\,\frac{\partial^2}{\partial\Delta^2}\left(\frac{\Delta-k_E^2}{(k_E^2+\Delta)^2}\right), \tag{131}$$

etc., *etc.* To take the Δ derivatives, we note that

$$\frac{\Delta-k_E^2}{(k_E^2+\Delta)^2} = \frac{1}{(\Delta+k_E^2)} - \frac{2k_E^2}{(\Delta+k_E^2)^2}\,, \tag{132}$$

hence

$$\begin{aligned}
\frac{\partial}{\partial\Delta}\left(\frac{\Delta-k_E^2}{(k_E^2+\Delta)^2}\right) &= \frac{-1}{(\Delta+k_E^2)^2} + \frac{4k_E^2}{(\Delta+k_E^2)^3} = \frac{3k_E^2-\Delta}{(\Delta+k_E^2)^3}\,,\\
\frac{\partial^2}{\partial\Delta^2}\left(\frac{\Delta-k_E^2}{(k_E^2+\Delta)^2}\right) &= \frac{+2}{(\Delta+k_E^2)^3} + \frac{-12k_E^2}{(\Delta+k_E^2)^4} = \frac{2\Delta-10k_E^2}{(\Delta+k_E^2)^4}\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{133}$$

Consequently, the first derivative of $\Sigma(p^2)$ is

$$\frac{d\Sigma_\Phi}{dp^2} = -4g^2\int_0^1 d\xi\,\xi(1-\xi)\times\int\frac{d^4k_E}{(2\pi)^4}\,\frac{3k_E^2-\Delta}{(\Delta+k_E^2)^3}\,. \tag{134}$$

In this formula, the integrand at large Euclidean momenta $k_E^2\gg\Delta$ behaves as $1/k_E^4$, so the integral suffers from a logarithmic UV divergence. But note that this is a milder UV divergence than the quadratic divergence of the integral (127) for the $\Sigma(p^2)$ itself!

Next, for the second derivative of $\Sigma(p^2)$ we have

$$\frac{d^2\Sigma_\Phi}{(dp^2)^2} = +4g^2 \int\limits_0^1 d\xi\, \xi^2(1-\xi)^2 \times \int \frac{d^4k_E}{(2\pi)^4} \frac{2\Delta - 10k_E^2}{(\Delta + k_E^2)^4}\,. \tag{135}$$

This time, the integrand at large Euclidean momenta behaves as $1/k_E^6$, so the integral converges in 4D. Thus,

$$\frac{d^2\Sigma_\Phi}{(dp^2)^2} \quad \text{is a finite function of } p^2. \tag{136}$$

From the first derivative's $d\Sigma_\Phi/dp^2$ point of view, the finite second derivative means that the divergence of the first derivative must be constant, *i.e.*, p^2–independent, thus

$$\frac{d\Sigma_\Phi}{dp^2} = \textit{divergent\_constant} \ + \ \textit{finite\_function}(p^2). \tag{137}$$

Similarly, integrating once again WRT to p^2, we obtain that the Σ_Φ itself has form

$$\Sigma_\Phi(p^2) \ = \ \textit{divergent\_constant}_1 \ + \ \textit{divergent\_constant}_2 \times p^2 \ + \ \textit{finite\_function}(p^2) \tag{138}$$

where

$$\textit{divergent\_constant}_1 \ = \ O(\Lambda^2) \quad \text{while} \quad \textit{divergent\_constant}_2 \ = \ O(\log\Lambda^2). \tag{139}$$

Later in class we shall learn that this is a general behavior of Σ_Φ for any scalar field in any renormalizable theory, and to all loop orders. In particular, your shall see this behavior in your homework set#15 in the $\lambda\Phi^4$ theory at the 2-loop level of perturbation theory.

Now that we know how the divergences of Σ_Φ depend on the scalar's momentum, let actually calculate the $\Sigma_\Phi(p^2)$ using dimensional regularization. This UV cutoff is going to miss the leading $O(\Lambda^2)$ divergence in the constant term in eq. (138), but it should get all the other features of the Σ_Φ. We start by rewriting the integrand in eq. (127) as

$$\frac{\Delta - k_E^2}{(k_E^2+\Delta)^2} \;=\; \frac{2\Delta}{(k_E^2+\Delta)^2} \;-\; \frac{1}{(k_E^2+\Delta)} \;=\; \int\limits_0^\infty dt\,(2\Delta\times t\;-\;1)\times e^{-t(k_E^2+\Delta)}. \tag{140}$$

Consequently, the momentum integral in D Euclidean dimensions becomes

$$\begin{aligned}
\mu^{4-D}\int\frac{d^Dk_E}{(2\pi)^D}\,\frac{\Delta-k_E^2}{(k_E^2+\Delta)^2} \;&=\; \mu^{4-D}\int\frac{d^Dk_E}{(2\pi)^D}\int\limits_0^\infty dt\,(2\Delta\times t\;-\;1)\times e^{-t(k_E^2+\Delta)} \\
&=\; \int\limits_0^\infty dt\,(2\Delta t-1)\,e^{-t\Delta}\times\mu^{4-D}\int\frac{d^Dk_E}{(2\pi)^D}\,e^{-tk_E^2} \\
&=\; \int\limits_0^\infty dt\,(2\Delta t-1)\,e^{-t\Delta}\times\mu^{4-D}(4\pi t)^{-D/2} \\
&=\; \mu^{4-D}(4\pi)^{-D/2}\int\limits_0^\infty dt\,e^{-t\Delta}\big(2\Delta t^{1-(D/2)}\;-\;t^{-(D/2)}\big).
\end{aligned} \tag{141}$$

On the last line here, the integrand for $t\to0$ behaves as $t^{-(D/2)}$, so the t-integral converges only for $D<2$. Consequently, we must evaluate this integral in a dimension lowered to $D<2$ and then analytically continue D up to 4 or rather $4-2\epsilon$ dimensions.

For $D<2$, the t integral on the last line of eq. (141) is a Γ-function integral, or rather

$$\begin{aligned}
\int\limits_0^\infty dt\,e^{-t\Delta}\big(2\Delta t^{1-(D/2)}\;-\;t^{-(D/2)}\big) \;&=\; 2\Delta\times\Gamma\big(2-\tfrac{D}{2}\big)\times\Delta^{(D/2)-2}\;-\;\Gamma\big(1-\tfrac{D}{2}\big)\times\Delta^{(D/2)-1} \\
&=\; \Delta^{(D/2)-1}\times\big[2\Gamma\big(2-\tfrac{D}{2}\big)\;-\;\Gamma\big(1-\tfrac{D}{2}\big)\big].
\end{aligned} \tag{142}$$

Moreover,

$$\Gamma\big(1-\tfrac{D}{2}\big) \;=\; \frac{\Gamma\big(2-\frac{D}{2}\big)}{1-\frac{D}{2}}\,, \tag{143}$$

hence

$$\big[2\Gamma\big(2-\tfrac{D}{2}\big)\;-\;\Gamma\big(1-\tfrac{D}{2}\big)\big] \;=\; \Gamma\big(2-\tfrac{D}{2}\big)\times\left[2\;-\;\frac{1}{1-\frac{D}{2}}\;=\;\frac{2D-2}{D-2}\right], \tag{144}$$

which has poles (in the complex D plane) at $D=2$ as well as at $D=4$, $D=6$, $D=8$, *etc.* As we saw earlier, a pole at $D < 2$ is the dimensional regularization's signal that the UV divergence is worse than logarithmic.

Altogether, we have in $D < 2$ dimensions

$$\Sigma_\Phi \;=\; 4g^2\mu^{4-D}(4\pi)^{-D/2}\,\Gamma\big(2-\tfrac{D}{2}\big)\frac{2D-2}{D-2}\times\int\limits_0^1 d\xi\,[\Delta(\xi)]^{(D/2)-1}. \tag{145}$$

Analytically continuing this formula to $D = 4 - 2\epsilon$ dimensions, we get

$$\Sigma_\Phi \;=\; \frac{g^2}{4\pi^2}\,\Gamma(\epsilon)\,\frac{6-4\epsilon}{2-2\epsilon}\times\int\limits_0^1 d\xi\,\Delta(\xi)\times\left(\frac{4\pi\mu^2}{\Delta(\xi)}\right)^{\epsilon}. \tag{146}$$

In the $\epsilon \to 0$ limit, we have

$$\Gamma(\epsilon) \;=\; \frac{1}{\epsilon}-\gamma_E+O(\epsilon),\quad \frac{6-4\epsilon}{2-2\epsilon} \;=\; 3+\epsilon+O(\epsilon^2),\quad \left(\frac{4\pi\mu^2}{\Delta(\xi)}\right)^{\epsilon} \;=\; 1+\epsilon\log\frac{4\pi\mu^2}{\Delta(\xi)}+O(\epsilon^2), \tag{147}$$

hence

$$\Sigma_\Phi(p^2) \;=\; \frac{3g^2}{4\pi^2}\int\limits_0^1 d\xi\,\Delta(\xi)\times\left(\frac{1}{\epsilon}-\gamma_E+\frac{1}{3}+\log\frac{4\pi\mu^2}{\Delta(\xi)}\right). \tag{148}$$

Note the divergent part of this formula:

$$\Sigma_\Phi^{\text{divergent}} \;=\; \frac{1}{\epsilon}\times\frac{3g^2}{4\pi^2}\times\int\limits_0^1 d\xi\,\big(\Delta(\xi) \;=\; m_f^2-\xi(1-\xi)p^2\big) \;=\; \frac{1}{\epsilon}\times\frac{3g^2}{4\pi^2}\times(m_f^2-\tfrac{1}{6}p^2). \tag{149}$$

This divergent part indeed has form $a + b \times p^2$ for some divergent constants a and b. For the b constant, the dimensional regularization correctly shows that

$$b \;=\; O(1/\epsilon)\quad\text{which corresponds to}\quad O(\log\Lambda^2), \tag{150}$$

while for the a constant the DR misses the leading $O(\Lambda^2)$ divergence and shows only the subleading logarithmic divergence

$$a \;\propto\; m_f^2\times\frac{1}{\epsilon}\;\to\; m_f^2\times\log\Lambda^2. \tag{151}$$

As to the finite part of $\Sigma_\Phi(p^2)$, in your homework set#15 you will see that it's real for $p^2<4m_f^2$ but becomes complex for $p^2>4m_f^2$. In particular, IF $m_s>2m_f$ — so that a scalar particle may decay into a fermion and an antifermion — then at $p^2=m_s^2$, the imaginary part of Σ_Φ is related to the scalar's decay rate as

$$\operatorname{Im}\Sigma_\Phi^{1\,\text{loop}}(p^2=m_s^2) \;=\; -m_s\Gamma^{\text{tree}}(\Phi\to\overline{\Psi}+\Psi). \tag{152}$$

This is the leading order in g^2 of the *optical theorem* for decay rates

$$\operatorname{Im}\Sigma_\Phi(p^2=M_s^2) \;=\; -M_s\Gamma_{\text{total}}(\Phi\to\text{anything}). \tag{153}$$

Finally,, consider the momentum derivative $d\Sigma_\Phi/dp^2$ — which we need to calculate the field strength renormalization factor Z_Φ. Taking the derivative of eq. (145) before taking the $D\to4$ limit, we have

$$\frac{d\Sigma_\Phi}{dp^2} \;=\; 4g^2\mu^{4-D}(4\pi)^{-D/2}\,\Gamma\bigl(2-\tfrac{D}{2}\bigr)\frac{D-1}{\frac{D}{2}-1}\times\int\limits_0^\infty d\xi\,\frac{\partial}{\partial p^2}[\Delta(\xi,p^2)]^{(D/2)-1} \tag{154}$$

where

$$\frac{\partial}{\partial p^2}\Delta^{(D/2)-1} \;=\; \bigl(\tfrac{D}{2}-1\bigr)\Delta^{(D/2)-2}\times\left(\frac{\partial\Delta}{\partial p^2}\;=\;-\xi(1-\xi)\right), \tag{155}$$

thus

$$\frac{d\Sigma_\Phi}{dp^2} \;=\; -4g^2\mu^{4-D}(4\pi)^{-D/2}\,(D-1)\Gamma\bigl(2-\tfrac{D}{2}\bigr)\times\int\limits_0^\infty d\xi\,\xi(1-\xi)\times[\Delta(\xi,p^2)]^{(D/2)-2}. \tag{156}$$

Note that in the complex D plane, this $d\Sigma_\Phi/dp^2$ has poles at $D=4$, $D=6$, $D=8$, *etc.*, but no pole at $D=2$ or any other $D<4$. This is the dimensional regularization's way of saying that the UV divergence of the $d\Sigma_\Phi/dp^2$ is purely logarithmic and we are not missing a quadratic (or other power-of-Λ) divergence.

Next, let $D = 4 - 2\epsilon$ and take the $\epsilon \to 0$ limit, thus

$$\begin{aligned}
4g^2\mu^{4-D}(4\pi)^{-D/2}\,(D-1)\Gamma\bigl(2-\tfrac{D}{2}\bigr)\times\Delta^{(D/2)-2} &= \\
&= \frac{4g^2}{16\pi^2}\,(3-2\epsilon)\Gamma(\epsilon)\times\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon} \\
&\longrightarrow \frac{3g^2}{4\pi^2}\,(1-\tfrac{2}{3}\epsilon)\left(\frac{1}{\epsilon}-\gamma_E+O(\epsilon)\right)\times\left(1+\epsilon\log\frac{4\pi\mu^2}{\Delta}+O(\epsilon^2)\right) \\
&= \frac{3g^2}{4\pi^2}\left(\frac{1}{\epsilon}-\gamma_E-\frac{2}{3}+\log\frac{4\pi\mu^2}{\Delta}+O(\epsilon)\right).
\end{aligned} \tag{157}$$

Neglecting the positive powers of ϵ in the $\epsilon \to 0$ limit and plugging this formula back into eq. (156), we arrive at

$$\frac{d\Sigma_\Phi}{dp^2} = -\frac{3g^2}{4\pi^2}\int_0^1 d\xi\,\xi(1-\xi)\times\left[\frac{1}{\epsilon}-\gamma_E-\frac{2}{3}+\left(\log\frac{4\pi\mu^2}{\Delta(\xi)} = \log\frac{4\pi\mu^2}{m_f^2}-\log\frac{\Delta(\xi)}{m_f^2}\right)\right]. \tag{158}$$

Using

$$\int_0^1 d\xi\,\xi(1-\xi) = \frac{1}{6} \tag{159}$$

we may rewrite this formula as

$$\frac{d\Sigma_\Phi^{1\,\mathrm{loop}}}{dp^2} = -\frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon}-\gamma_E-\frac{2}{3}+\log\frac{4\pi\mu^2}{m_f^2}-I(p^2/m_f^2)\right). \tag{160}$$

where

$$\frac{I(p^2/m_f^2)}{6} = \int_0^1 d\xi\,\xi(1-\xi)\,\log\frac{\Delta = m_f^2-\xi(1-\xi)p^2}{m_f^2}\,. \tag{161}$$

The derivative (160) evaluated at $p^2 = M_s^2$ determines the scalar field's strength factor

$$Z_\Phi = \Bigl|\langle 1\text{ scalar particle}|\,\hat{\Phi}\,|\Omega\rangle\Bigr|^2 \tag{162}$$

according to

$$\frac{1}{Z_\Phi} = 1 - \left.\frac{d\Sigma_\Phi}{dp^2}\right|_{p^2=M_s^2} . \tag{163}$$

Thus, at the one-loop level

$$\frac{1}{Z_\Phi} = 1 + \frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon} - \gamma_E - \frac{2}{3} + \log\frac{4\pi\mu^2}{m_f^2} - I(M_s^2/m_f^2)\right) + O(g^4), \tag{164}$$

and therefore

$$Z_\Phi = 1 - \frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon} - \gamma_E - \frac{2}{3} + \log\frac{4\pi\mu^2}{m_f^2} - I(M_s^2/m_f^2)\right) + O(g^4). \tag{165}$$