Renormalizability and Dimensional Analysis

In these notes I shall explain the relation between energy dimensionalities of the coupling constants of a quantum field theory and between super-renormalizability, renormalizability, or non-renormalizability of the theory.

Let's start with the basic dimensional analysis. In the $\hbar = c = 1$ units, all quantities are measured in units of energy to some power. For example $[m] = [p^\mu] = E^{+1}$ while $[x^\mu] = E^{-1}$, where $[m]$ stands for the *dimensionality* of the mass rather than the mass itself, and ditto for the $[p^\mu]$, $[x^\mu]$, *etc.* The action

$$S = \int d^4x\, \mathcal{L}$$

is dimensionless (in $\hbar \neq 1$ units, $[S] = \hbar$), so the Lagrangian of a 4D field theory has dimensionality $[\mathcal{L}] = E^{+4}$.

Dimensionalities — also called the *canonical dimensions* — of the quantum fields follow from their free Lagrangians.

For example, a scalar field $\Phi(x)$ has

$$\mathcal{L}_{\text{free}} = \tfrac{1}{2}\partial_\mu \Phi\, \partial^\mu \Phi - \tfrac{1}{2} m^2 \Phi^2, \tag{1}$$

so $[\mathcal{L}] = E^{+4}$, $\left[m^2\right] = E^{+2}$, and $[\partial_\mu] = E^{+1}$ imply $[\Phi] = E^{+1}$. Likewise, the EM field has

$$\mathcal{L}_{\text{free}}^{\text{EM}} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \Longrightarrow \quad [F_{\mu\nu}] = E^{+2}, \tag{2}$$

and since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the $A_\nu(x)$ field has dimension

$$[A_\nu] = [F_{\mu\nu}] \,\big/\, [\partial_\mu] = E^{+1}. \tag{3}$$

In fact, all the *bosonic* fields in 4D spacetime have canonical dimensions E^{+1} because their kinetic terms are quadratic in ∂_μ(field). On the other hand, the fermionic fields like the Dirac field $\Psi(x)$ have dimensionality $[\Psi] = E^{+3/2}$. Indeed, the kinetic terms in the free

Dirac Lagrangian

$$\mathcal{L}_{\text{free}} \;=\; \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{4}$$

involve two fermionic fields Ψ and $\overline{\Psi}$ but only one derivative ∂_μ. Consequently, $[\mathcal{L}] = E^{+4}$ implies $\left[\overline{\Psi}\Psi\right] = E^{+3}$ and hence $[\Psi] = \left[\overline{\Psi}\right] = E^{+3/2}$. Similarly, all other types of fermionic fields in 4D have canonical dimension $E^{+3/2}$.

In QFTs in other spacetime dimensions $d \neq 4$, similar arguments show that the bosonic fields such as scalars and vectors have canonical dimension

$$[\Phi] \;=\; [A_\nu] \;=\; E^{+(d-2)/2} \tag{5}$$

while the fermionic fields have canonical dimension

$$[\Psi] \;=\; E^{+(d-1)/2}. \tag{6}$$

In perturbation theory, dimensionality of coupling parameters such as λ in $\lambda\Phi^4$ theory or e in QED follows from the field's canonical dimensions. For example, in a 4D scalar theory with Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi \;-\; \tfrac{1}{2}m^2\Phi^2 \;-\sum_{n\geq 3}\frac{C_n}{n!}\,\Phi^n, \tag{7}$$

the coupling C_n of the Φ^n term has dimensionality

$$[C_n] \;=\; [\mathcal{L}] \Big/ [\Phi]^n \;=\; E^{4-n}. \tag{8}$$

In particular, the cubic coupling C_3 has positive energy dimension E^{+1}, the quartic coupling $\lambda = C_4$ is dimensionless, while all the higher-power couplings have negative energy dimensions E^{negative}. Note how the sign of the coupling's energy dimension matches the renormalizability of the theory: the super-renormalizable coupling κ has a positive energy dimension, the renormalizable coupling λ is dimensionless, and the non-renormalizable couplings C_n for $n > 4$ have negative energy dimensions. This is an example of a general rule:

- All couplings of a renormalizable theory must have non-negative energy dimensions.
- If all the couplings of a theory have strictly positive energy dimensions, then the theory is super-renormalizable.
- But if any copuling of a theory has a negative energy dimension, then the theory is non-renormalizable, even if it also have other couplings of non-negative energy dimensions.

To see how this works, consider a generic interaction term in the Lagrangian of some QFT. In general such term is a product of some coupling constant g and several fields or their derivatives. Let n_b be the number of bosonic fields in this product, n_f the number of fermionic fields, and n_d the number of spacetime derivatives ∂_μ acting on all these fields. Consequently,

$$[\text{field product}] \; = \; E^{n_b+\frac{3}{2}n_f+n_d}, \tag{9}$$

and since the entire interaction term must have dimensionality E^{+4} — same as the entire Lagrangian — the coupling constant g must have dimensionality

$$[g] \; = \; E^{\Delta} \quad \text{for} \quad \Delta \; = \; 4 \; - \; n_b \; - \; \tfrac{3}{2}n_f \; - \; n_d\,. \tag{10}$$

In general, a QFT may have several coupling constants, and each coupling has its own energy dimension Δ according to eq. (10).

Next, consider a Feynman diagram for some QFT. Let the diagram have L loops, P_b bosonic propagators, P_f fermionic propagators, and V vertices of all kinds, so the diagram evaluates to

$$\int d^{4L}q \prod(\text{propagators}) \times \prod(\text{vertices}). \tag{11}$$

Consider the superficial degree of divergence $\mathcal{D}$ of such a diagram. At large momenta q, each bosonic propagator behaves as $1/q^2$ while each fermionic propagator behaves as $1/q$. The vertices may also be momentum-dependent: if the interaction term in the Lagrangian involves n_d derivatives of fields, then the corresponding vertex includes n_d power of momenta, so for large q it grows as q^{+n+d}. Altogether, the momentum integral (11) behaves as

$$\int d^{4L}q\, \frac{1}{q^{2P_b+P_f}} \times \prod_{v}^{\text{vertices}} q^{+n_d(v)}, \tag{12}$$

so its superficial degree of divergence is

$$\mathcal{D} \;=\; 4L \;-\; 2P_b \;-\; P_f \;+\; \sum_{v=1}^{V} n_d(v). \tag{13}$$

Now let's rework this formula using basic graph theory. By the Euler theorem

$$L \;-\; P_{\text{net}} \;+\; V \;=\; 1 \quad\Longrightarrow\quad L \;=\; 1 \;+\; P_b \;+\; P_f \;-\; V, \tag{14}$$

hence

$$\mathcal{D} \;=\; 4 \;+\; (4-2=2)\times P_b \;+\; (4-1=3)\times P_f \;+\; \sum_{v=1}^{V}(n_d-4). \tag{15}$$

Also, counting the line ends — bosonic or fermionic — we obtain

$$2P_b \;+\; E_b \;=\; \sum_{v} n_b(v), \tag{16}$$

$$2P_f \;+\; E_f \;=\; \sum_{v} n_f(v), \tag{17}$$

and hence

$$2P_b \;+\; 3P_f \;=\; \sum_{v=1}^{V}(n_b+\tfrac{3}{2}n_f) \;-\; E_b \;-\; \tfrac{3}{2}E_f\,. \tag{18}$$

Consequently, eq. (15) becomes

$$\mathcal{D} \;=\; 4 \;-\; E_b \;-\; \tfrac{3}{2}E_f \;+\; \sum_{v=1}^{V}(n_b+\tfrac{3}{2}n_f+n_d-4). \tag{19}$$

Note that the combinations $(n_b+\frac{3}{2}n_f+n_d-4)$ we sum over the vertices are precisely (minus) the energy dimensions of the corresponding couplings, *cf.* eq. (10). Thus, we arrive at the key relation

$$\mathcal{D} = 4 - E_b - \tfrac{3}{2}E_f - \sum_{v=1}^{V} \Delta(g_v). \tag{20}$$

between the couplings' energy dimensions and the divergence degrees of the Feynman diagrams.

The rules relating couplings' dimensions Δ to the renormalizability of the QFT in question follow from eq. (20):

- If all the couplings of the theory have strictly positive dimensions Δ, then only a finite number of Feynman diagrams for the theory may have $\mathcal{D} \geq 0$ and hence suffer from the overall UV divergence. All the rest of the diagrams are either UV-finite of have divergent sub-diagrams — but once the subgraph divergence is canceled by an in-situ counterterm, the overall diagram becomes finite. And that's what makes the theory in question super-renormalizable.

- If some couplings of the theory are dimensionless ($\Delta = 0$) while other have $\Delta > 0$, then the theory has an infinite number of diagrams with $\mathcal{D} \geq 0$ and therefore divergent. But all such diagrams must have $E_b + \frac{3}{2}E_f \leq 4$, which means that there is only a finite number of divergent *amplitudes*. Consequently, all the UV divergences can be canceled by a finite set of counterterms, but the coefficients of such counterterms must be adjusted order-by-order in perturbation theory at all loop orders. And that's what makes the theory in question renormalizable.

- Finally, if a theory has a coupling with a negative dimension Δ, then the theory has an infinite number of divergent amplitudes. Indeed, for any given numbers of external bosonic and fermionic legs, eq. (20) allows for $\mathcal{D} \geq 0$ provided the diagram includes enough vertices with $\Delta < 0$. Consequently, the theory needs an infinite set of counterterms to cancel all such divergences, and that's what makes it non-renormalizable.

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At this point we know the significance of the coupling's dimensions

$$\Delta \;=\; 4 \;-\; n_b \;-\; \tfrac{3}{2}n_f \;-\; n_d\,, \tag{10}$$

let's classify the renormalizable ($\Delta = 0$) and the super-renormalizable ($\Delta > 0$) couplings of 4D field theories. Since any physical interaction term involves at least 3 fields (otherwise, it would be a part of the free Lagrangian), it follows that the only way to get $\Delta > 0$ is to have $n_b = 3$, $n_f = 0$, and $n_d = 0$, — in other words, boson^3 without ∂_μ derivatives. Likewise, there are only 3 ways to get a renormalizable coupling with $\Delta = 0$, namely boson^4, boson^2 $\times$ ∂ boson, and boson $\times$ fermion^2. All other combinations of fields lead to non-renormalizable couplings with $\Delta < 0$.

In terms of more specific types of fields and couplings, *there is only one kind of a super-renormalizable coupling*, namely the 3-scalar coupling

$$-\frac{\kappa}{6}\,\Phi^3, \quad \text{or for multiple fields} \quad -\sum_{i,j,k}\frac{\kappa_{ijk}}{6}\Phi_i\Phi_j\Phi_k\,. \tag{21}$$

Also, there are only 5 kinds of renormalizable couplings:

1. The 4-scalar coupling

$$-\frac{\lambda}{24}\,\Phi^4, \quad \text{or for multiple fields} \quad -\sum_{i,j,k,\ell}\frac{\lambda_{ijk\ell}}{24}\,\Phi_i\Phi_j\Phi_k\Phi_\ell\,. \tag{22}$$

2. Gauge couplings of vectors to charged scalars

$$-iqA^\mu \times (\Phi^*\partial_\mu\Phi \;-\; \Phi\partial_\mu\Phi^*) \;+\; q^2A_\mu A^\mu \times \Phi^*\Phi \;\subset\; D_\mu\Phi^*\, D^\mu\Phi, \tag{23}$$

or for non-abelian gauge symmetries

$$-igA^{a\mu} \times \Big(\Phi^\dagger T^a\partial_\mu\Phi \;-\; \partial_\mu\Phi^\dagger T^a\Phi\Big) \;+\; g^2A^a_\mu A^{b\mu} \times \Phi^\dagger T^aT^b\Phi \;\subset\; D_\mu\Phi^\dagger\, D^\mu\Phi. \tag{24}$$

3. Non-abelian gauge couplings between the vector fields

$$-gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} \;-\; \frac{g^2}{4}\,f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e} \;\subset\; -\frac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu a}. \tag{25}$$

4. Gauge couplings of vectors to charged fermions,

$$-qA^\mu \times \overline{\Psi}\gamma_\mu\Psi \quad \text{or} \quad -gA^{a\mu} \times \overline{\Psi}\gamma_\mu T^a\Psi \quad \subset \quad \overline{\Psi}(i\gamma_\mu D^\mu)\Psi. \tag{26}$$

   If the fermions are massless and chiral, we may also have

$$-gA^a_\mu \times \overline{\Psi}\gamma^\mu \frac{1 \mp \gamma^5}{2} T^a\Psi, \tag{27}$$

   or in the Weyl fermion language

$$-gA^a_\mu \times \psi^\dagger_L \bar{\sigma}_\mu T^a \psi_L \quad \text{or} \quad -gA^a_\mu \times \psi^\dagger_R \sigma_\mu T^a \psi_R\,.$$

5. Yukawa couplings of scalars to fermions,

$$-y\Phi_1 \times \overline{\Psi}\Psi \quad \text{or} \quad -iy\Phi_2 \times \overline{\Psi}\gamma^5\Psi. \tag{28}$$

   If parity is conserved, then Φ_1 should be a true scalar and Φ_2 a pseudo-scalar.

— And this is it! All other coupling types are non-renormalizable in 4 spacetime dimensions.

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In other spacetime dimensions $d \neq 3+1$, a coupling involving n_b bosonic fields, n_f fermionic fields, and n_d derivatives has dimensionality

$$\begin{aligned} \Delta(d) \;&=\; d \;-\; n_b \times \frac{d-2}{2} \;-\; n_f \times \frac{d-1}{2} \;-\; n_d \\ &=\; \left(4 \;-\; n_b \;-\; \tfrac{3}{2}n_f \;-\; n_d\right) \;-\; \frac{n_b+n_f-2}{2} \times (d-4) \\ &=\; \Delta(d=4) \;-\; \frac{n_b+n_f-2}{2} \times (d-4). \end{aligned} \tag{29}$$

Since all interactions involve three or more fields, thus $n_b+n_f \geq 3$, the dimensionality of any particular coupling always decreases with spacetime dimension d. Consequently, there are more (super)renormalizable couplings with $\Delta \geq 0$ in lower dimensions $d = 2+1$ or $d = 1+1$ but fewer such couplings in higher dimensions $d > 3+1$. In particular,

- *In $d \geq 6+1$ dimensions all couplings have $\Delta < 0$ and there are no renormalizable couplinds at all!*
- In $d = 5+1$ dimensions there is a unique $\Delta = 0$ coupling $(\kappa/6)\Phi^3$, while all the other couplings have $\Delta < 0$. Consequently, the only renormalizable theories are scalar theories with cubic potentials,

$$\mathcal{L} \;=\; \sum_i \left(\tfrac{1}{2}(\partial_\mu \Phi_a)^2 \;-\; \tfrac{1}{2} m_i^2 \Phi_a^2\right) \;-\; \tfrac{1}{6} \sum_{i,j,k} \mu_{ijk} \Phi_i \Phi_j \Phi_k \,. \tag{30}$$

  However, while such theories are perturbatively OK, they do not have stable vacua since a cubic potential is always unbounded from below.
- In $d = 4+1$ dimensions, the $(\kappa/6)\Phi^3$ coupling has positive $\Delta = +\frac{1}{2}$ while all the other couplings have negative energy dimensions. Hence, the scalar theories (30) are super-renormalizable (but non-perturbatively sick), while all other interactive QFTs are non-renormalizable.

⋆ The bottom line is, *in $d > 3+1$ dimensions there are no renormalizable theories with stable vacua.*

On the other hand, in lower dimensions $d = 2+1$ or $d = 1+1$ there are many more (super)renormalizable $\Delta \geq 0$. In particular, in $d = 2+1$ dimensions such couplings include:

◦ Scalar couplings $(C_n/n!)\Phi^n$ up to $n = 6$;

◦ Gauge and Yukawa couplings like in 4D;

◦ Yukawa-like couplings $\tilde{y}\Phi^2 \times \overline{\Psi}\Psi$ involving 2 scalars;

∗ Chern–Simons couplings of non-abelian gauge fields to each other, and some other exotic couplings, never mind the details.

Finally, *in $d = 1+1$ dimensions there are infinite numbers of renormalizable and even super-renormalizable couplings.* Indeed, for $d = 1+1$ the bosonic fields have energy dimension E^0, so Δ of a coupling does not depend on the number n_b of bosonic fields it involves but only on the numbers of derivatives and fermionic fields,

$$\Delta \;=\; 2 - \; n_d \;-\; \tfrac{1}{2} n_f \,. \tag{31}$$

Consequently, all scalar potentials $V(\Phi)$ — including $C_n\Phi^n$ terms for any n, and even the non-polynomial potentials — have $\Delta = +2$, so any $V(\Phi)$ potential is super-renormalizable in 2D. Likewise, all Yukawa-like couplings $\Phi^n\overline{\Psi}\Psi$ have $\Delta = +1$, so we may have terms like $y_{IJ}(\Phi) \times \overline{\Psi}^I \Psi^J$ for any functions $y_{IJ}(\Phi)$.

At the $\Delta = 0$ level, we have renormalizable field-dependent kinetic terms

$$\mathcal{L}_{\text{kin}} \;=\; \tfrac{1}{2} g_{ij}(\phi) \times \partial^\mu \phi^i \, \partial_\mu \phi^j \tag{32}$$

with any Riemannian metrics $g_{ij}(\phi)$ for the non-linear scalar field space, as well as a whole bunch of fermionic terms with arbitrary scalar-dependent coefficients,

$$\begin{aligned}\mathcal{L}_\Psi \;\supset\; \tfrac{1}{4} g_{IJ}(\Phi) \times \overline{\Psi}^I \gamma^\mu \left(i\,\overrightarrow{\partial_\mu} - i\,\overleftarrow{\partial_\mu} \right) \Psi^J \;+\; \Gamma_{IJk}(\Phi) \times \partial_\mu \Phi^k \times \overline{\Psi}^I \gamma^\mu \Psi^J \\ +\; \tfrac{1}{2} R_{IJKL}(\Phi) \times \overline{\Psi}^I \gamma^\mu \Psi^J \times \overline{\Psi}^K \gamma_\mu \Psi^L.\end{aligned} \tag{33}$$

In addition, there are gauge couplings with arbitrary scalar-dependent $g_{\text{gauge}}(\Phi)$, chiral couplings to Weyl or Majorana-Weyl fermions, *etc.*, *etc.* In String Theory, many of these couplings show up the context of the 2D field theory on the world sheet of the string.