Renormalizability and Dimensional Analysis

In these notes I shall explain the relation between energy dimensionalities of the coupling constants of a quantum field theory and between super-renormalizability, renormalizability, or non-renormalizability of the theory.

Let’s start with the basic dimensional analysis. In the $\hbar = c = 1$ units, all quantities are measured in units of energy to some power. For example $[m] = [p^\mu] = E^{+1}$ while $[x^\mu] = E^{-1}$, where $[m]$ stands for the dimensionality of the mass rather than the mass itself, and ditto for the $[p^\mu], [x^\mu]$, etc. The action

$$ S = \int d^4x \mathcal{L} $$

is dimensionless (in $\hbar \neq 1$ units, $[S] = \hbar$), so the Lagrangian of a 4D field theory has dimensionality $[\mathcal{L}] = E^{+4}$.

Dimensionalities — also called the canonical dimensions — of the quantum fields follow from their free Lagrangians.

For example, a scalar field $\Phi(x)$ has

$$ \mathcal{L}_{\text{free}} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2, $$

so $[\mathcal{L}] = E^{+4}, [m^2] = E^{+2}$, and $[\partial_\mu] = E^{+1}$ imply $[\Phi] = E^{+1}$. Likewise, the EM field has

$$ \mathcal{L}_{\text{free}}^{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \implies [F_{\mu\nu}] = E^{+2}, $$

and since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the $A_\nu(x)$ field has dimension

$$ [A_\nu] = [F_{\mu\nu}] / [\partial_\mu] = E^{+1}. $$

In fact, all the bosonic fields in 4D spacetime have canonical dimensions $E^{+1}$ because their kinetic terms are quadratic in $\partial_\mu$(field). On the other hand, the fermionic fields like the Dirac field $\Psi(x)$ have dimensionality $[\Psi] = E^{+3/2}$. Indeed, the kinetic terms in the free
Dirac Lagrangian

\[ \mathcal{L}_{\text{free}} = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \]  

(4)

involve two fermionic fields \( \Psi \) and \( \overline{\Psi} \) but only one derivative \( \partial_\mu \). Consequently, \( [\mathcal{L}] = E^4 \) implies \( [\overline{\Psi}\Psi] = E^3 \) and hence \( [\Psi] = [\overline{\Psi}] = E^{3/2} \). Similarly, all other types of fermionic fields in 4D have canonical dimension \( E^{3/2} \).

In QFTs in other spacetime dimensions \( d \neq 4 \), similar arguments show that the bosonic fields such as scalars and vectors have canonical dimension

\[ [\Phi] = [A_\nu] = E^{+(d-2)/2} \]  

(5)

while the fermionic fields have canonical dimension

\[ [\Psi] = E^{+(d-1)/2}. \]  

(6)

In perturbation theory, dimensionality of coupling parameters such as \( \lambda \) in \( \lambda \Phi^4 \) theory or \( e \) in QED follows from the field’s canonical dimensions. For example, in a 4D scalar theory with Lagrangian

\[ \mathcal{L} = \frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \sum_{n \geq 3} \frac{C_n}{n!} \Phi^n, \]  

(7)

the coupling \( C_n \) of the \( \Phi^n \) term has dimensionality

\[ [C_n] = [\mathcal{L}] / [\Phi]^n = E^{4-n}. \]  

(8)

In particular, the cubic coupling \( C_3 \) has positive energy dimension \( E^1 \), the quartic coupling \( \lambda = C_4 \) is dimensionless, while all the higher-power couplings have negative energy dimensions \( E^{\text{negative}} \). Note how the sign of the coupling’s energy dimension matches the renormalizability of the theory: the super-renormalizable coupling \( \kappa \) has a positive energy dimension, the renormalizable coupling \( \lambda \) is dimensionless, and the non-renormalizable couplings \( C_n \) for \( n > 4 \) have negative energy dimensions. This is an example of a general rule:
• All couplings of a renormalizable theory must have non-negative energy dimensions.

• If all the couplings of a theory have strictly positive energy dimensions, then the theory is super-renormalizable.

• But if any coupling of a theory has a negative energy dimension, then the theory is non-renormalizable, even if it also have other couplings of non-negative energy dimensions.

To see how this works, consider a generic interaction term in the Lagrangian of some QFT. In general such term is a product of some coupling constant $g$ and several fields or their derivatives. Let $n_b$ be the number of bosonic fields in this product, $n_f$ the number of fermionic fields, and $n_d$ the number of spacetime derivatives $\partial_\mu$ acting on all these fields. Consequently,

$$[\text{field product}] = E^{n_b + \frac{3}{2} n_f + n_d},$$

(9)

and since the entire interaction term must have dimensionality $E^{+4}$ — same as the entire Lagrangian — the coupling constant $g$ must have dimensionality

$$[g] = E^\Delta \quad \text{for} \quad \Delta = 4 - n_b - \frac{3}{2} n_f - n_d.$$  

(10)

In general, a QFT may have several coupling constants, and each coupling has its own energy dimension $\Delta$ according to eq. (10).

Next, consider a Feynman diagram for some QFT. Let the diagram have $L$ loops, $P_b$ bosonic propagators, $P_f$ fermionic propagators, and $V$ vertices of all kinds, so the diagram evaluates to

$$\int d^4q \prod (\text{propagators}) \times \prod (\text{vertices}).$$

(11)

Consider the superficial degree of divergence $D$ of such a diagram. At large momenta $q$, each bosonic propagator behaves as $1/q^2$ while each fermionic propagator behaves as $1/q$. The vertices may also be momentum-dependent: if the interaction term in the Lagrangian involves $n_d$ derivatives of fields, then the corresponding vertex includes $n_d$ power of momenta,
so for large $q$ it grows as $q^{n+d}$. Altogether, the momentum integral (11) behaves as

$$\int d^4Lq \frac{1}{q^{2P_b+P_f}} \times \prod_v q^{\nu_d(v)},$$

so its superficial degree of divergence is

$$\mathcal{D} = 4L - 2P_b - P_f + \sum_{v=1}^V n_d(v).$$

Now let’s rework this formula using basic graph theory. By the Euler theorem

$$L - P_{net} + V = 1 \implies L = 1 + P_b + P_f - V,$$

hence

$$\mathcal{D} = 4 + (4 - 2 = 2) \times P_b + (4 - 1 = 3) \times P_f + \sum_{v=1}^V (n_d - 4).$$

Also, counting the line ends — bosonic or fermionic — we obtain

$$2P_b + E_b = \sum_v n_b(v),$$

$$2P_f + E_f = \sum_v n_f(v),$$

and hence

$$2P_b + 3P_f = \sum_{v=1}^V (n_b + \frac{3}{2}n_f) - E_b - \frac{3}{2}E_f.$$

Consequently, eq. (15) becomes

$$\mathcal{D} = 4 - E_b - \frac{3}{2}E_f + \sum_{v=1}^V (n_b + \frac{3}{2}n_f + n_d - 4).$$

Note that the combinations $(n_b + \frac{3}{2}n_f + n_d - 4)$ we sum over the vertices are precisely (minus) the energy dimensions of the corresponding couplings, cf. eq. (10). Thus, we arrive at the
key relation

\[ \mathcal{D} = 4 - E_b - \frac{3}{2}E_f - \sum_{v=1}^{V} \Delta(g_v). \] (20)

between the couplings’ energy dimensions and the divergence degrees of the Feynman diagrams.

The rules relating couplings’ dimensions \( \Delta \) to the renormalizability of the QFT in question follow from eq. (20):

- If all the couplings of the theory have strictly positive dimensions \( \Delta \), then only a finite number of Feynman diagrams for the theory may have \( \mathcal{D} \geq 0 \) and hence suffer from the overall UV divergence. All the rest of the diagrams are either UV-finite of have divergent sub-diagrams — but once the subgraph divergence is canceled by an in-situ counterterm, the overall diagram becomes finite. And that’s what makes the theory in question super-renormalizable.

- If some couplings of the theory are dimensionless (\( \Delta = 0 \)) while other have \( \Delta > 0 \), then the theory has an infinite number of diagrams with \( \mathcal{D} \geq 0 \) and therefore divergent. But all such diagrams must have \( E_b + \frac{3}{2}E_f \leq 4 \), which means that there is only a finite number of divergent amplitudes. Consequently, all the UV divergences can be canceled by a finite set of counterterms, but the coefficients of such counterterms must be adjusted order-by-order in perturbation theory at all loop orders. And that’s what makes the theory in question renormalizable.

- Finally, if a theory has a coupling with a negative dimension \( \Delta \), then the theory has an infinite number of divergent amplitudes. Indeed, for any given numbers of external bosonic and fermionic legs, eq. (20) allows for \( \mathcal{D} \geq 0 \) provided the diagram includes enough vertices with \( \Delta < 0 \). Consequently, the theory needs an infinite set of counterterms to cancel all such divergences, and that’s what makes it non-renormalizable.

\* \* \*
At this point we know the significance of the coupling’s dimensions

\[ \Delta = 4 - n_b - \frac{3}{2} n_f - n_d, \]  

(10)

let’s classify the renormalizable (\(\Delta = 0\)) and the super-renormalizable (\(\Delta > 0\)) couplings of

4D field theories. Since any physical interaction term involves at least 3 fields (otherwise, it would be a part of the free Lagrangian), it follows that the only way to get \(\Delta > 0\) is to have \(n_b = 3\), \(n_f = 0\), and \(n_d = 0\), — in other words, \(\text{boson}^3\) without \(\partial_\mu\) derivatives. Likewise, there are only 3 ways to get a renormalizable coupling with \(\Delta = 0\), namely \(\text{boson}^4\), \(\text{boson}^2 \times \partial \, \text{boson}\), and \(\text{boson} \times \text{fermion}^2\). All other combinations of fields lead to non-renormalizable couplings with \(\Delta < 0\).

In terms of more specific types of fields and couplings, there is only one kind of a super-renormalizable coupling, namely the 3-scalar coupling

\[-\frac{\kappa}{6} \Phi^3, \quad \text{or for multiple fields} \quad - \sum_{i,j,k} \frac{\kappa_{ijk}}{6} \Phi_i \Phi_j \Phi_k.\]  

(21)

Also, there are only 5 kinds of renormalizable couplings:

1. The 4-scalar coupling

\[-\frac{\lambda}{24} \Phi^4, \quad \text{or for multiple fields} \quad - \sum_{i,j,k,\ell} \frac{\lambda_{ijkl}}{24} \Phi_i \Phi_j \Phi_k \Phi_\ell.\]  

(22)

2. Gauge couplings of vectors to charged scalars

\[-i q A^\mu \times (\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) + q^2 A^\mu A^\mu \times \Phi^* \Phi \subset D_\mu \Phi^* D^\mu \Phi,\]  

(23)

or for non-abelian gauge symmetries

\[-i g A^{a\mu} \times \left( \Phi^T a \partial_\mu \Phi - \partial_\mu \Phi^T a \Phi \right) + g^2 A^a A^b \times \Phi^T a T^b \Phi \subset D_\mu \Phi^T D^\mu \Phi.\]  

(24)

3. Non-abelian gauge couplings between the vector fields

\[-g f^{abc} (\partial_\mu A^a_\mu) A^{tb} A^{nu} - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} \subset -\frac{1}{4} F_{\mu \nu}^a F^{\mu \nu a}.\]  

(25)
4. Gauge couplings of vectors to charged fermions,

\[-gA^\mu \times \overline{\Psi} \gamma_\mu \Psi \quad \text{or} \quad -gA^a_\mu \times \overline{\Psi} \gamma_\mu T^a \Psi \subset \overline{\Psi} (i\gamma_\mu D^\mu) \Psi. \tag{26}\]

If the fermions are massless and chiral, we may also have

\[-gA^a_\mu \times \overline{\Psi} \gamma_\mu \frac{1 \mp \gamma^5}{2} T^a \Psi, \tag{27}\]

or in the Weyl fermion language

\[-gA^a_\mu \times \psi_L^\dagger \sigma_\mu \psi_L \quad \text{or} \quad -gA^a_\mu \times \psi_R^\dagger \sigma_\mu T^a \psi_R.\]

5. Yukawa couplings of scalars to fermions,

\[-y\Phi_1 \times \overline{\Psi} \Psi \quad \text{or} \quad -iy\Phi_2 \times \overline{\Psi} \gamma^5 \Psi. \tag{28}\]

If parity is conserved, then \(\Phi_1\) should be a true scalar and \(\Phi_2\) a pseudo-scalar.

— And this is it! All other coupling types are non-renormalizable in 4 spacetime dimensions.

\[\ast \ast \ast \]

In other spacetime dimensions \(d \neq 3 + 1\), a coupling involving \(n_b\) bosonic fields, \(n_f\) fermionic fields, and \(n_d\) derivatives has dimensionality

\[
\Delta(d) = d - n_b \times \frac{d - 2}{2} - n_f \times \frac{d - 1}{2} - n_d
\]

\[
= \left(4 - n_b - \frac{3}{2}n_f - n_d\right) - \frac{n_b + n_f - 2}{2} \times (d - 4) \tag{29}
\]

\[
= \Delta(d = 4) - \frac{n_b + n_f - 2}{2} \times (d - 4).
\]

Since all interactions involve three or more fields, thus \(n_b + n_f \geq 3\), the dimensionality of any particular coupling always decreases with spacetime dimension \(d\). Consequently, there are more (super)renormalizable couplings with \(\Delta \geq 0\) in lower dimensions \(d = 2 + 1\) or \(d = 1 + 1\) but fewer such couplings in higher dimensions \(d > 3 + 1\). In particular,
• In $d \geq 6 + 1$ dimensions all couplings have $\Delta < 0$ and there are no renormalizable couplings at all!

• In $d = 5 + 1$ dimensions there is a unique $\Delta = 0$ coupling $(\kappa/6)\Phi^3$, while all the other couplings have $\Delta < 0$. Consequently, the only renormalizable theories are scalar theories with cubic potentials,

$$\mathcal{L} = \sum_i \left( \frac{1}{2} (\partial_\mu \Phi_a)^2 - \frac{1}{2} m_i^2 \Phi_a^2 \right) - \frac{1}{6} \sum_{i,j,k} \mu_{ijk} \Phi_i \Phi_j \Phi_k. \quad (30)$$

However, while such theories are perturbatively OK, they do not have stable vacua since a cubic potential is always unbounded from below.

• In $d = 4 + 1$ dimensions, the $(\kappa/6)\Phi^3$ coupling has positive $\Delta = +\frac{1}{2}$ while all the other couplings have negative energy dimensions. Hence, the scalar theories (30) are super-renormalizable (but non-perturbatively sick), while all other interactive QFTs are non-renormalizable.

* The bottom line is, in $d > 3 + 1$ dimensions there are no renormalizable theories with stable vacua.

On the other hand, in lower dimensions $d = 2 + 1$ or $d = 1 + 1$ there are many more (super)renormalizable $\Delta \geq 0$. In particular, in $d = 2 + 1$ dimensions such couplings include:

○ Scalar couplings $(C_n/n!)\Phi^n$ up to $n = 6$;

○ Gauge and Yukawa couplings like in 4D;

○ Yukawa-like couplings $\tilde{y}\Phi^2 \times \overline{\Psi}\Psi$ involving 2 scalars;

* Chern–Simons couplings of non-abelian gauge fields to each other, and some other exotic couplings, never mind the details.

Finally, in $d = 1 + 1$ dimensions there are infinite numbers of renormalizable and even super-renormalizable couplings. Indeed, for $d = 1+1$ the bosonic fields have energy dimension $E^0$, so $\Delta$ of a coupling does not depend on the number $n_b$ of bosonic fields it involves but only on the numbers of derivatives and fermionic fields,

$$\Delta = 2 - n_d - \frac{1}{2} n_f. \quad (31)$$

Consequently, all scalar potentials $V(\Phi)$ — including $C_n\Phi^n$ terms for any $n$, and even the
non-polynomial potentials — have $\Delta = +2$, so any $V(\Phi)$ potential is super-renormalizable in 2D. Likewise, all Yukawa-like couplings $\Phi^a \overline{\Psi} \Psi$ have $\Delta = +1$, so we may have terms like $y_{IJ}(\Phi) \times \overline{\Psi}^I \Psi^J$ for any functions $y_{IJ}(\Phi)$.

At the $\Delta = 0$ level, we have renormalizable field-dependent kinetic terms

$$L_{\text{kin}} = \frac{1}{2} g_{ij}(\phi) \times \partial^\mu \phi^i \partial_\mu \phi^j$$

with any Riemannian metrics $g_{ij}(\phi)$ for the non-linear scalar field space, as well as a whole bunch of fermionic terms with arbitrary scalar-dependent coefficients,

$$L_{\Psi} \supset \frac{1}{4} g_{IJ}(\Phi) \times \overline{\Psi}^I \gamma^\mu \left( i \overleftarrow{\partial}_\mu - i \overrightarrow{\partial}_\mu \right) \Psi^J + \Gamma_{IJK}(\Phi) \times \partial_\mu \Phi^K \times \overline{\Psi}^I \gamma^\mu \Psi^J + \frac{1}{2} R_{IJKL}(\Phi) \times \overline{\Psi}^I \gamma_\mu \Psi^J \times \overline{\Psi}^K \gamma_\mu \Psi^L.$$  (33)

In addition, there are gauge couplings with arbitrary scalar-dependent $g_{\text{gauge}}(\Phi)$, chiral couplings to Weyl or Majorana-Weyl fermions, etc., etc. In String Theory, many of these couplings show up the context of the 2D field theory on the world sheet of the string.