Functional Quantization

In quantum mechanics of one or several particles, transitions between different quantum states are governed by the matrix elements of the evolution operator

$$\langle\text{out}|\,\hat{U}(t_{\text{out}}-t_{\text{in}})\,|\text{in}\rangle \;=\int\!d^N\vec{x}_{\text{in}}\!\int\!d^N\vec{x}_{\text{out}}\,\Psi^*_{\text{out}}(\vec{x}_{\text{out}})\times U(t_{\text{out}},\vec{x}_{\text{out}};t_{\text{in}},\vec{x}_{\text{in}})\times\Psi_{\text{in}}(\vec{x}_{\text{in}}),\quad(1)$$

where $\vec{x}$ stand for the coordinates of all the particles in the system. Last lecture (*cf.* my notes) we learned that the evolution kernel here obtains from the path integral

$$U(t_{\text{out}},\vec{x}_{\text{out}};t_{\text{in}},\vec{x}_{\text{in}}) \;=\iint\limits_{\vec{x}(t_{\text{in}})=\vec{x}_{\text{in}}}^{\vec{x}(t_{\text{out}})=\vec{x}_{\text{out}}}\mathcal{D}'[\vec{x}(t)]\,e^{iS[\vec{x}(t)]}\qquad(2)$$

for

$$S[\vec{x}(t)] \;=\int\limits_{t_{\text{in}}}^{t_{\text{out}}}L\,dt,\qquad L \;=\sum_{i=1}^{N}\frac{m_i}{2}\,\dot{x}_i^2\,-\,V(\vec{x}).\qquad(3)$$

Plugging this formula into eq. (1), we may combine the path integral over $\vec{x}(t)$ with the integrals over the initial and final values of $\vec{x}$ to

$$\langle\text{out}|\,\hat{U}(t_{\text{out}}-t_{\text{in}})\,|\text{in}\rangle \;=\iint\mathcal{D}[\vec{x}(t)]\,\exp(iS[\vec{x}(t)])\times\Psi^*_{\text{out}}(\vec{x}(t_{\text{out}}))\,\Psi_{\text{in}}(\vec{x}(t_{\text{in}})).\qquad(4)$$

without any boundary conditions for the $\vec{x}(t)$. Note that in this path integral the path of each coordinate $x_i(t)$ is independent of all other coordinates, thus

$$\iint\mathcal{D}[\vec{x}(t)] \;=\prod_{i=1}^{N}\iint\mathcal{D}[x_i(t)].\qquad(5)$$

Let's generalize formula (4) to the quantum field theory. In classical field theory, a finite set $\vec{x}$ of dynamical variables x_i becomes a field $\Phi(\mathbf{x})$ where $\mathbf{x}$ is now a continuous label rather than a dynamical variable of its own. Thus, a set of all the particles' paths $x_i(t)$ becomes a *field configuration* $\Phi(\mathbf{x},t)$, with action functional

$$S[\Phi(\mathbf{x},t)] \ = \int\limits_{t_{\rm in}}^{t_{\rm out}} dt \int d^3\mathbf{x}\, \mathcal{L} \tag{6}$$

where the Lagrangian density is

$$\mathcal{L} \ = \ \frac{1}{2}\partial_\mu\Phi\, \partial^\mu\Phi \ - \ \frac{m^2}{2}\,\Phi^2 \ - \ \frac{\lambda}{24}\,\Phi^4 \tag{7}$$

for a scalar field, or something more complicated for other kinds of fields.

In the quantum field theory, the path integral (5) generalizes to the *functional integral* over the field configurations

$$\iint \mathcal{D}[\Phi(\mathbf{x},t)] \tag{8}$$

A careful definition of this functional integral involves discretization of both time and space to a 4D lattice of $N = TV/a^4$ spacetime points x_n and then taking a zero-lattice-spacing limit

$$\iint \mathcal{D}[\Phi(\mathbf{x},t)] \ = \ \lim_{a\to 0} \text{Normalization\_factor} \times \prod_n \int d\Phi(x_n). \tag{9}$$

I shall address this issue later in class, perhaps next lecture. For now, let's focus on using the functional integral.

The QFT analogue of eq. (4) for the transition matrix element is

$$\langle\text{out}|\,\hat U(t_{\rm out}-t_{\rm in})\,|\text{in}\rangle \ = \iint \mathcal{D}[\Phi(\mathbf{x},t)]\ \exp(iS[\Phi(\mathbf{x},t)]) \times \Psi^*_{\rm out}[\Phi(\mathbf{x})\ @t_{\rm out}]\ \Psi_{\rm in}[\Phi(\mathbf{x})\ @t_{\rm in}]. \tag{10}$$

Note that in the field theory's Hilbert space, the wave-functions $\Psi_{\rm in}$ and $\Psi_{\rm out}$ become wave-functionals of the entire 3-space field configuration $\Phi(\mathbf{x})$ for a fixed time $t = t_{\rm in}$ or $t = t_{\rm out}$. In terms of such wave-functionals, the $\langle\text{out}|\text{in}\rangle$ bracket is given by a 3-space analogue of the functional integrals over $\Phi(\mathbf{x})$ at a fixed time,

$$\langle\text{out}|\text{in}\rangle \ = \iint \mathcal{D}[\Phi(\mathbf{x})]\ \Psi^*_{\rm out}[\Phi(\mathbf{x})] \times \Psi_{\rm in}[\Phi(\mathbf{x})]. \tag{11}$$

Now consider a matrix element of a quantum field $\hat\Phi(\mathbf{x}_1,t_1)$ at some point $\mathbf{x}_1$ and time t_1 between an initial state $|\text{in}\rangle$ defined at an earlier time $t_{\rm in} < t_1$ and a final state $\langle\text{out}|$ defined at a later time $t_{\rm out} > t_1$. The Heisenberg and the Schrödinger pictures of this matrix element are related as

$$\langle{\rm in}|\,\hat\Phi_H(\mathbf{x}_1,t_1)\,|{\rm out}\rangle \;=\; \langle{\rm out}|\,\hat U(t_{\rm out}-t_1)\hat\Phi_S(\mathbf{x}_1)\hat U(t_1-t_{\rm in})\,|{\rm in}\rangle\,, \tag{12}$$

where the evolution operator and its matrix elements obtain from the functional integrals (10). Using the $|\Phi\rangle$ basis in which all $\hat\Phi_S(x)$ are diagonal, we obtain

$$\begin{aligned}
&\langle{\rm out}|\hat\Phi_H(\mathbf{x}_1,t_1)\,|{\rm in}\rangle \;=\\
&\quad= \iiint\mathcal{D}[\Phi_1(\mathbf{x})]\;\langle{\rm out}|\,\hat U(t_{\rm out}-t_1)\,|\Phi_1\rangle\times\Phi_1(\mathbf{x}_1)\times\langle\Phi_1|\,\hat U(t_1-t_{\rm in})\,|{\rm in}\rangle\\
&\quad= \iiint\mathcal{D}[\Phi_1(\mathbf{x})]\,\Phi_1(\mathbf{x}_1)\times\\
&\qquad\times\iiint\limits_{\Phi(\mathbf{x},t_1)=\Phi_1(\mathbf{x})}\mathcal{D}[\Phi(x,t)\,{\rm for}\,t_1<t<t_{\rm out}]\;\Psi^*_{\rm out}[\Phi(\mathbf{x})\,@t_{\rm out}]\times\exp\bigl(iS[\Phi]\,{\rm from}\,t_1\,{\rm to}\,t_{\rm out}\bigr)\times\\
&\qquad\times\iiint^{\Phi(\mathbf{x},t_1)=\Phi_1(\mathbf{x})}\mathcal{D}[\Phi(x,t)\,{\rm for}\,t_{\rm in}<t<t_1]\;\exp\bigl(iS[\Phi]\,{\rm from}\,t_{\rm in}\,{\rm to}\,t_1\bigr)\times\Psi_{\rm in}[\Phi(\mathbf{x})\,@t_{\rm in}]\\
&\quad= \iiint\mathcal{D}[\Phi(\mathbf{x},t)\text{ for the whole time}]\;\Psi^*_{\rm out}[\Phi(\mathbf{x})\,@t_{\rm out}]\;\Psi_{\rm in}[\Phi(\mathbf{x})\,@t_{\rm in}]\times\\
&\qquad\qquad\times\exp\bigl(iS[\Phi]\text{ for the whole time}\bigr)\times\Phi(\mathbf{x}_1,t_1).
\end{aligned}\tag{13}$$

Next, consider a matrix element of a product of two fields,

$$\langle{\rm out}|\,\hat\Phi_H(\mathbf{x}_2,t_2)\hat\Phi_H(\mathbf{x}_1,t_1)\,|{\rm in}\rangle \;=\; \langle out|\,\hat U(t_{\rm out},t_2)\hat\Psi_S(\mathbf{x}_2)\hat U(t_2,t_1)\hat\Phi_S(\mathbf{x}_1)\hat U(t_1,t_{\rm in})\,|{\rm in}\rangle\,, \tag{14}$$

which in the $|\Phi\rangle$ basis where all the $\hat\Phi_S(\mathbf{x})$ operators are diagonal becomes

$$\iiint\mathcal{D}[\Phi_2(\mathbf{x})]\iiint\mathcal{D}[\Phi_1(\mathbf{x})]\;\langle{\rm out}|\,\hat U(t_{\rm out},t_2)\,|\Phi_2\rangle\,\Phi_2(\mathbf{x}_2)\,\langle\Phi_2|\,\hat U(t_2,t_1)\,|\Phi_1\rangle\,\Phi_1(\mathbf{x}_1)\,\langle\Phi_1|\,\hat U(t_1,t_{\rm in})\,|{\rm in}\rangle. \tag{15}$$

For $t_{\rm out} > t_2 > t_1 > t_{\rm in}$, all the evolution operators here work forward in time, so we may use the functional integrals to calculate their matrix elements. Thus,

$$
\begin{aligned}
\langle\text{out}|\hat{\Phi}_H(\mathbf{x}_2,t_2)\hat{\Phi}_H(\mathbf{x}_1,t_1)\,|\text{in}\rangle\ &= \\
= \iint\!\mathcal{D}[\Phi_2(\mathbf{x})] \iint\!\mathcal{D}[\Phi_1(\mathbf{x})]\,\Phi_2(\mathbf{x}_2)\Phi_1(\mathbf{x}_1)\,\times \\
\times \iint\limits_{\Phi(\mathbf{x},t_2)=\Phi_2(\mathbf{x})} \mathcal{D}[\Phi(x,t)\,\text{for}\,t_2<t<t_{\text{out}}]\,\Psi^*_{\text{out}}[\Phi(\mathbf{x})\,@t_{\text{out}}]\times\exp\bigl(iS[\Phi]\,\text{from}\,t_2\,\text{to}\,t_{\text{out}}\bigr)\,\times \\
\times \iint\limits_{\substack{\Phi(\mathbf{x},t_2)=\Phi_2(\mathbf{x})\\ \Phi(\mathbf{x},t_1)=\Phi_1(\mathbf{x})}} \mathcal{D}[\Phi(x,t)\,\text{for}\,t_1<t<t_2]\,\exp\bigl(iS[\Phi]\,\text{from}\,t_1\,\text{to}\,t_2\bigr)\,\times \\
\times \iint\limits^{\Phi(\mathbf{x},t_1)=\Phi_1(\mathbf{x})} \mathcal{D}[\Phi(x,t)\,\text{for}\,t_{\text{in}}<t<t_1]\,\Psi_{\text{in}}[\Phi(\mathbf{x})\,@t_{\text{in}}]\times\exp\bigl(iS[\Phi]\,\text{from}\,t_{\text{in}}\,\text{to}\,t_1\bigr)\,\times \\
= \iint\!\mathcal{D}[\Phi(\mathbf{x},t)\ \text{for the whole time}]\ \Psi^*_{\text{out}}[\Phi(\mathbf{x})\,@t_{\text{out}}]\ \Psi_{\text{in}}[\Phi(\mathbf{x})\,@t_{\text{in}}]\,\times \\
\times\exp\bigl(iS[\Phi]\ \text{for the whole time}\bigr)\times\Phi(\mathbf{x}_2,t_2)\Phi(\mathbf{x}_1,t_1).
\end{aligned}
\tag{16}
$$

Note however that this formula works only for $t_2 > t_1$ so all the evolution operators work forward in time. It would not work for $t_1 > t_2$, unless we re-order the quantum fields in the matrix element, thus

$$
\begin{aligned}
\langle\text{out}|\hat{\Phi}_H(\mathbf{x}_1,t_1)\hat{\Phi}_H(\mathbf{x}_2,t_2)\,|\text{in}\rangle\ &= \\
= \iint\!\mathcal{D}[\Phi(\mathbf{x},t)\ \text{for the whole time}]\ \Psi^*_{\text{out}}[\Phi(\mathbf{x})\,@t_{\text{out}}]\ \Psi_{\text{in}}[\Phi(\mathbf{x})\,@t_{\text{in}}]\,\times \\
\times\exp\bigl(iS[\Phi]\ \text{for the whole time}\bigr)\times\Phi(\mathbf{x}_1,t_1)\Phi(\mathbf{x}_2,t_2).
\end{aligned}
\tag{17}
$$

Inside the functional integrals on the RH sides of eq. (16) and (17), the fields $\Phi(\mathbf{x}_1,t_1)$ and $\Phi(\mathbf{x}_2,t_2)$ are classical variables so their order does not matter. But on the LH sides, the fields are quantum operators, their order does matter, and the formulae (16) and (17) work only when those operators are in time order. In other words, the functional integral automatically puts the operators in the time order,

$$
\begin{aligned}
\iint\!\mathcal{D}[\Phi(\mathbf{x},t)\ \text{for the whole time}]\ \exp\bigl(iS[\Phi]\ \text{for the whole time}\bigr)\,\times \\
\times\ \Psi^*_{\text{out}}[\Phi(\mathbf{x})\,@t_{\text{out}}]\,\Psi_{\text{in}}[\Phi(\mathbf{x})\,@t_{\text{in}}]\ \times\ \Phi(\mathbf{x}_1,t_1)\Phi(\mathbf{x}_2,t_2) \\
=\ \langle\text{out}|\ \mathbf{T}\hat{\Phi}_H(\mathbf{x}_1,t_1)\hat{\Phi}_H(\mathbf{x}_2,t_2)\,|\text{in}\rangle\,.
\end{aligned}
\tag{18}
$$

Likewise, similar functional integrals involving several fields yield matrix elements of *time-*

ordered products of those fields,

$$\int\!\!\!\int \mathcal{D}[\Phi(\mathbf{x},t) \text{ for the whole time}]\, \exp\bigl(iS[\Phi] \text{ for the whole time}\bigr) \times$$
$$\times\, \Psi^*_{\rm out}[\Phi(\mathbf{x})\,@t_{\rm out}]\,\Psi_{\rm in}[\Phi(\mathbf{x})\,@t_{\rm in}]\,\times \Phi(\mathbf{x}_1,t_1)\cdots\Phi(\mathbf{x}_n,t_n)$$
$$= \langle{\rm out}|\,\mathbf{T}\,\hat\Phi_H(\mathbf{x}_1,t_1)\cdots\hat\Phi_H(\mathbf{x}_n,t_n)\,|{\rm in}\rangle\,. \tag{19}$$

Now, let's take the time interval between the in and out states to infinity in both directions, or rather let's take the limit

$$t_{\rm out}\ \to\ +\infty\times(1-i\epsilon),\quad t_{\rm in}\ \to\ -\infty\times(1-i\epsilon). \tag{20}$$

Also, let's assume that both the incoming and the outgoing states have the same quantum numbers as the vacuum state $|\Omega\rangle$. Then, as we saw back in January — *cf.* class notes on correlation functions, — in the limit (20) we get

$$\begin{aligned} \langle{\rm out}|\,e^{-i\hat H t_{\rm out}}\ &\to\ \langle\Omega|\times\text{c-number factor},\\ e^{+i\hat H t_{\rm in}}\,|{\rm in}\rangle\ &\to\ |\Omega\rangle\times\text{c-number factor}, \end{aligned} \tag{21}$$

regardless of the details of the initial and final states. Consequently, on the LHS of eq. (19) we get the *correlation functions of* n *quantum fields*

$$G_n(x_1,x_2,\ldots,x_n)\ =\ \langle\Omega|\,\mathbf{T}\hat\Phi_H(x_1)\cdots\hat\Phi_H(x_n)\,|\Omega\rangle \tag{22}$$

while on the RHS we may just as well let

$$\Psi_{\rm in}[\Phi(\mathbf{x})\,@t_{\rm in}]\ \longrightarrow\ \text{const}\quad\text{and}\quad \Psi^*_{\rm out}[\Phi(\mathbf{x})\,@t_{\rm out}]\ \longrightarrow\ \text{const}. \tag{23}$$

Thus, we arrive at the functional integral formula for the correlation functions,

$$G_n(x_1,x_2,\ldots,x_n)\ =\ \text{const}\times\int\!\!\!\int\mathcal{D}[\Phi(x)]\,\exp\left(iS[\Phi]\ =\ i\int d^4x\,\mathcal{L}\right)\times\Phi(x_1)\times\cdots\times\Phi(x_n), \tag{24}$$

where the action integral $S=\int d^4x\,\mathcal{L}$ for each field configuration $\Phi(x)$ is taken over the whole Minkowski space time.

Similar formulae apply to correlation functions in other quantum field theories like QED. In all such theories, the correlation functions on the LHS are automatically time-ordered by the 'magic' of functional integration. Also, as long as the action integral of the theory is Lorentz invariant, the functional integrals (24) are manifestly Lorentz invariant.*

We do not know the constant factor in front of the functional integral (24) and we do not know the overall normalization of the functional integral. But we can get rid of both of these unknown factors by taking ratios of correlation functions. In particular, using

$$G_0() \;=\; \langle\Omega|\Omega\rangle \;=\; 1 \tag{25}$$

we can write all the other correlation functions as normalization-independent ratios

$$\begin{aligned} G_n(x_1, x_2, \ldots, x_n) \;&=\; \frac{\langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle}{\langle\Omega|\Omega\rangle} \\ &=\; \frac{\displaystyle\iint\!\mathcal{D}[\Phi(x)]\,\exp\left(i\int d^4x\,\mathcal{L}\right)\times\Phi(x_1)\times\cdots\times\Phi(x_n)}{\displaystyle\iint\!\mathcal{D}[\Phi(x)]\,\exp\left(i\int d^4x\,\mathcal{L}\right)}\,. \end{aligned} \tag{26}$$

In a few pages, we shall learn how to calculate such ratios in perturbation theory and re-derive the Feynman rules. But before we do the perturbation theory, let's see how functional integrals work for the free scalar field.

* Strictly speaking, the limit (20) implies that all the time coordinates x^0 of all fields in (24) live on the slightly tilted time axis that runs from $-\infty\times(1-i\epsilon)$ to $+\infty\times(1-i\epsilon)$. Consequently, the action has a complexified version of the Lorentz symmetry, but after analytic continuation of all the correlation functions back to the real axis, we recover the usual Lorentz symmetry $SO(3,1)$. Alternatively, we may analytically continue to the imaginary times; this gives us $SO(4)$ symmetry in the Euclidean spacetime.

Functional Integrals for the Free Scalar Field

The free scalar field has a quadratic action functional

$$S[\Phi(x)] \;=\int\! d^4x \left(\mathcal{L} \;=\; \tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi \;-\; \tfrac{1}{2}m^2\Phi^2\right) \;=\; -\frac{1}{2}\int\! d^4x\,\Phi(x)\,(\partial^2+m^2)\Phi(x). \qquad (27)$$

Hence, the functional integral

$$Z \;=\iint\!\mathcal{D}[\Phi(x)]\;\exp\bigl(iS[\Phi(x)]\bigr) \qquad (28)$$

is a kind of a Gaussian integral, generalization of the ordinary Gaussian integral

$$Z(A) \;=\int\! d\xi_1\int\! d\xi_2\cdots\int\! d\xi_N\;\exp\left(-\frac{1}{2}\sum_{j,k=1}^{N}A_{jk}\xi_j\xi_k\right) \qquad (29)$$

to a continuous family of variables, $\{\xi_j\}\to\mathcal{D}[\Phi(x)]$.

Lemma 1: The ordinary Gaussian integral (29) evaluates to

$$Z(A) \;=\; \frac{(2\pi)^{N/2}}{\sqrt{\det(A)}}\,. \qquad (30)$$

Note: A_{jk} is a symmetric $N\times N$ matrix, real or complex. To make the integral (29) converge, the real part $\operatorname{Re}A_{jk}$ of the A matrix should be positive definite.

Proof: Any symmetric matrix can be diagonalized, thus

$$\sum_{jk}A_{jk}\xi_j\xi_k \;=\; \sum_k B_k\eta_k^2 \qquad (31)$$

where $\eta_1,\ldots,\eta_N$ are some independent linear combinations of the $\xi_1,\ldots,\xi_N$. Changing integration variables from the ξ_k to η_k carries a Jacobian

$$J \;=\; \left|\det\left(\frac{\partial\xi_j}{\partial\eta_k}\right)\right|\;; \qquad (32)$$

by linearity of the $\xi \to \eta$ transform, this Jacobian is constant. Therefore,

$$\begin{aligned} Z &= J \times \int d^N\eta \exp\left(-\frac{1}{2}\sum_k B_k\eta_k^2\right) \\ &= J \times \prod_{k=1}^{N} \int d\eta_k \exp\left(-\frac{B_k}{2}\,\eta_k^2\right) \\ &= J \times \prod_{k=1}^{N} \sqrt{\frac{2\pi}{B_k}} \;=\; \frac{(2\pi)^{N/2} \times J}{\sqrt{\det(B)}} \end{aligned} \tag{33}$$

where B is the diagonal matrix $B_{jk} = \delta_{jk} \times B_k$. This matrix is related to A according to

$$B \;=\; \left(\frac{\partial\xi}{\partial\eta}\right)^{\top} A \left(\frac{\partial\xi}{\partial\eta}\right), \tag{34}$$

hence

$$\det(B) \;=\; \det(A) \times \det{}^2\left(\frac{\partial\xi}{\partial\eta}\right) \;=\; \det(A) \times J^2 \tag{35}$$

and therefore

$$Z \;=\; (2\pi)^{N/2} \times \frac{J}{\sqrt{\det(B)}} \;=\; \frac{(2\pi)^{N/2}}{\sqrt{\det(A)}}\,. \tag{30}$$

Lemma 2 concerns the ratios of Gaussian integrals for the same matrix A:

$$\frac{\int d^N\xi\ \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right) \times \xi_k\xi_\ell}{\int d^N\xi\ \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right)} \;=\; \left(A^{-1}\right)_{k\ell}\,. \tag{36}$$

Proof:

$$\int d^N\xi \exp\left(-\tfrac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right) \times \xi_k\xi_\ell \;=\; -2\frac{\partial}{\partial A_{k\ell}} \int d^N\xi \exp\left(-\tfrac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right), \tag{37}$$

hence

$$\frac{\int d^N\xi\ \exp\left(-\frac{1}{2}\sum_{ij}A_{ij}\xi_i\xi_j\right)\times\xi_k\xi_\ell}{\int d^N\xi\ \exp\left(-\frac{1}{2}\sum_{ij}A_{ij}\xi_i\xi_j\right)} = -2\frac{\partial}{\partial A_{k\ell}}\log\left[\int d^N\xi\exp\left(-\frac{1}{2}\sum_{ij}A_{ij}\xi_i\xi_j\right)\right]$$

$$= -2\frac{\partial}{\partial A_{k\ell}}\log\frac{(2\pi)^{N/2}}{\sqrt{\det(A)}} \tag{38}$$

$$= +\frac{\partial}{\partial A_{k\ell}}\log\det(A)$$

$$= \left(A^{-1}\right)_{k\ell}.$$

Both lemmas apply to the Gaussian functional integrals such as (28). For example, consider the free scalar propagator $G_2(x,y)$. Eq. (26) gives us a formula for this propagator in terms of a ratio of two Gaussian functional integrals

$$G_2(x,y) = \frac{\iiint\mathcal{D}[\Phi(x)]\ \exp\left(-\frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right)\times\Phi(x)\Phi(y)}{\iiint\mathcal{D}[\Phi(x)]\ \exp\left(-\frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right)}. \tag{39}$$

The role of the matrix A here is played by the differential operator $i(\partial^2+m^2)$. This operator is purely anti-hermitian, which corresponds to a purely imaginary matrix A. To make the Gaussian integrals converge, we add an infinitesimal real part, $A_{ij}\to A_{ij}+\epsilon\delta_{ij}$, or $i(\partial^2+m^2)\to i(\partial^2+m^2)+\epsilon$. Consequently, lemma 2 tells us that

$$G_2(x,y) = \langle x|\,\frac{1}{i(\partial^2+m^2)+\epsilon}\,|y\rangle = \int\frac{d^4p}{(2\pi)^4}\,\frac{ie^{-ip(x-y)}}{p^2-m^2+i\epsilon}. \tag{40}$$

Note that the rule $A\to A+\epsilon$ we use to make the Gaussian integral converge for an imaginary (or anti-hermitian) matrix A immediately leads us to the Feynman propagator (40) rather than some other Green's function.

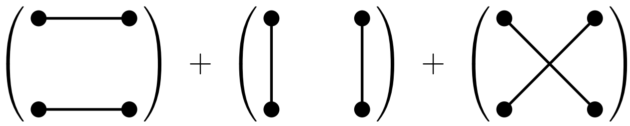
To obtain correlation functions G_n for $n>2$ we need analogues of Lemma 2 for Gaussian integrals involving more than two ξs outside the exponential. For example, for four ξ's we have

$$\frac{\int d^N\xi\ \exp\left(-\frac{1}{2}\sum_{ij}A_{ij}\xi_i\xi_j\right)\times\xi_k\xi_\ell\xi_m\xi_n}{\int d^N\xi\ \exp\left(-\frac{1}{2}\sum_{ij}A_{ij}\xi_i\xi_j\right)} = \tag{41}$$

$$= \left(A^{-1}\right)_{k\ell}\times\left(A^{-1}\right)_{mn} + \left(A^{-1}\right)_{km}\times\left(A^{-1}\right)_{\ell n} + \left(A^{-1}\right)_{kn}\times\left(A^{-1}\right)_{\ell m}.$$

Applying this formula to the Gaussian functional integral for the free scalar fields, we obtain for the 4–point function

$$G_4(x,y,z,w) \;=\; G_2(x-y)\times G_2(z-w) \;+\; G_2(x-z)\times G_2(y-w) \;+\; G_2(x-w)\times G_2(y-z) \tag{42}$$

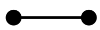
$=$  .

Likewise, for higher even numbers of ξs,

$$\frac{\int d^N\xi\, \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right)\times \xi_{(1)}\cdots\xi_{(N)}}{\int d^N\xi\, \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right)} \;=\; \sum_{\text{pairings}}\left(\prod_{\text{pairs}}\left(A^{-1}\right)_{\text{index pair}}\right) \tag{43}$$

and similarly

$$G_N^{\text{free}}(x_1,\ldots,x_N) \;=\; \frac{\iiint\mathcal{D}[\Phi(x)]\, \exp\left(-\frac{i}{2}\int d^4x\, \Phi(\partial^2+m^2)\Phi\right)\times\Phi(x_1)\cdots\Phi(x_N)}{\iiint\mathcal{D}[\Phi(x)]\, \exp\left(-\frac{i}{2}\int d^4x\, \Phi(\partial^2+m^2)\Phi\right)} \tag{44}$$

$= \sum_{\text{pairings}}\left(\prod_{\text{pairs}}\right.$  $\left.\right)$.

Feynman Rules for the Interacting Field

A self-interacting scalar field has action

$$S[\Phi(x)] \;=\; S_{\text{free}}[\Phi(x)] \;-\; \frac{\lambda}{4!}\int d^4z\, \Phi^4(z). \tag{45}$$

Expanding *the exponential* e^{iS} in powers of the coupling λ, we have

$$\exp\bigl(iS[\Phi(x)]\bigr) \;=\; \exp\bigl(iS_{\text{free}}[\Phi(x)]\bigr)\times\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\lambda}{4!}\int d^4z\, \Phi^4(z)\right)^n. \tag{46}$$

This expansion gives us the perturbation theory for the functional integral of the interacting field,

$$
\begin{aligned}
&\iint\mathcal{D}[\Phi(x)]\,\exp\bigl(iS[\Phi(x)]\bigr)\times\Phi(x_1)\cdots\Phi(x_k) \;= \\
&\quad= \iint\mathcal{D}[\Phi(x)]\,\exp\left(iS_{\text{free}}[\Phi(x)] \;=\; \frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right)\times \\
&\qquad\qquad \times\Phi(x_1)\cdots\Phi(x_k)\times\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\lambda}{4!}\int d^4z\,\Phi^4(z)\right)^n \\
&\quad= \sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{n!\,(4!)^n}\int d^4z_1\cdots\int d^4z_n \\
&\qquad \iint\mathcal{D}[\Phi(x)]\,\exp\left(\frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right)\times\Phi(x_1)\cdots\Phi(x_k)\times\Phi^4(z_1)\cdots\Phi^4(z_n) \\
&\qquad \langle\!\langle\text{using eq. (44) for the free-field functional integrals}\rangle\!\rangle \\
&\quad= \left[\iint\mathcal{D}[\Phi(x)]\,\exp\left(\frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right)\right]\times \\
&\qquad \times\sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{n!\,(4!)^n}\int d^4z_1\cdots\int d^4z_n\sum_{\text{pairings}}\left(\text{products of }\frac{k+4n}{2}\text{ free propagators}\right) \\
&\quad= \left[\iint\mathcal{D}[\Phi(x)]\,\exp\left(\frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right)\right]\times \\
&\qquad \times\sum_{n=0}^{\infty}\sum\Bigl(\text{Feynman diagrams with } k \text{ external and } n \text{ internal vertices}\Bigr).
\end{aligned}
\tag{47}
$$

This is how the Feynman rules emerge from the functional integral formalism.

At this stage, we are summing over all the Feynman diagrams, connected or disconnected, and even the vacuum bubbles are allowed. Also, the functional integral (47) carries an overall factor

$$
Z_0 \;=\; \iint\mathcal{D}[\Phi(x)]\,\exp\left(\frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right) \tag{48}
$$

which multiplies the whole perturbation series. Fortunately, this pesky factor — as well as all the vacuum bubbles — cancel out when we divide eq. (47) by a similar functional integral for

$k = 0$ to get the correlation function G_k. Indeed,

$$\begin{aligned} G_k(x_1,\ldots,x_k) &= \frac{\iiint \mathcal{D}[\Phi(x)]\,\exp\bigl(iS[\Phi(x)]\bigr) \times \Phi(x_1)\cdots\Phi(x_k)}{\iiint \mathcal{D}[\Phi(x)]\,\exp\bigl(iS[\Phi(x)]\bigr)} \\ &= \frac{\cancel{Z_0} \times \sum \text{all Feynman diagrams with } k \text{ external vertices}}{\cancel{Z_0} \times \sum \text{all Feynman diagrams without external vertices}} \\ &= \sum \text{Feynman diagrams with } k \text{ external vertices and no vacuum bubbles.} \end{aligned} \tag{49}$$

Generating Functional

In the functional integral formalism there is a compact way of summarizing all the *connected* correlation functions in terms of a single generating functional. Let's add to the scalar field's Lagrangian $\mathcal{L}$ a linear source term,

$$\mathcal{L}(\Phi,\partial\Phi) \;\to\; \mathcal{L}(\Phi,\partial\Phi) \;+\; J\times\Phi \tag{50}$$

where $J(x)$ is an arbitrary function of x. Now let's calculate the partition function for $\Phi(x)$ as a functional of the source $J(x)$,

$$\begin{aligned} Z[J(x)] &= \iiint \mathcal{D}[\Phi(x)]\,\exp\left(i\int d^4x(\mathcal{L}+J\Phi)\right) \\ &= \iiint \mathcal{D}[\Phi(x)]\,\exp\bigl(iS[\Phi(x)]\bigr) \times \exp\left(i\int d^4x\,J(x)\Phi(x)\right). \end{aligned} \tag{51}$$

To make use of this partition function, we need functional (*i.e.*, variational) derivatives $\delta/\delta J(x)$. Here is how they work:

$$\frac{\delta}{\delta J(x)}\int J(y)\Phi(y)d^4y \;=\; \Phi(x), \tag{52}$$

$$\frac{\delta}{\delta J(x)}\int \partial_\mu J(y)\times A^\mu(y)d^4y \;=\; -\partial_\mu A^\mu(x), \tag{53}$$

$$\frac{\delta}{\delta J(x)}\exp\left(i\int J(y)\Phi(y)d^4y\right) \;=\; i\Phi(x)\times\exp\left(i\int J(y)\Phi(y)d^4y\right), \tag{54}$$

etc., *etc.* Applying these derivatives to the partition function (51), we have

$$\frac{\delta}{\delta J(x)}\,Z[J] \;=\iint\!\mathcal{D}[\Phi(y)]\,\exp\left(i\int d^4y(\mathcal{L}+J\Phi)\right)\times i\Phi(x) \tag{55}$$

and similarly

$$\frac{\delta}{\delta J(x_1)}\,\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_k)}\,Z[J] \;=\iint\!\mathcal{D}[\Phi(y)]\,\exp\left(i\int d^4y(\mathcal{L}+J\Phi)\right)\times i^k\Phi(x_1)\Phi(x_2)\cdots\Phi(x_k). \tag{56}$$

Thus, the partition function (51) *generates* all the functional integrals of the scalar theory:

$$\forall k,\ \iint\!\mathcal{D}[\Phi(y)]\,\exp\left(i\int d^4y\mathcal{L}\right)\times\Phi(x_1)\cdots\Phi(x_k) \;=\; (-i)^k\,\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_k)}\,Z[J]\bigg|_{J(y)\equiv 0}. \tag{57}$$

Consequently, all the correlation functions (49) follows from the derivatives of the $Z[J]$ according to

$$G_k(x_1,\ldots,x_k) \;=\; \frac{(-i)^k}{Z[J]}\,\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_k)}\,Z[J]\bigg|_{J(y)\equiv 0}. \tag{58}$$

In terms of the Feynman diagrams, the correlation functions (58) contain both connected and disconnected graphs; only the vacuum bubbles are excluded. To obtain the connected correlation functions

$$G_k^{\rm conn}(x_1,\ldots,x_k) \;=\; \sum\text{connected Feynman diagrams only} \tag{59}$$

we should take the $\delta/\delta J(x)$ derivatives of $\log Z[J]$ rather than $Z[J]$ itself,

$$G_k^{\rm conn}(x_1,\ldots,x_k) \;=\; (-i)^k\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_k)}\,\log Z[J(y)]\bigg|_{J(y)\equiv 0}. \tag{60}$$

In other words, $\log Z[J(x)]$ is the *generating functional* for for all the connected correlation functions of the quantum field theory.

Proof: For the sake of generality, let's allow for $G_1(x) = \langle\Phi(x)\rangle \neq 0$. For $k = 1$, $G_1^{\rm conn} \equiv G_1$ and eq. (60) is equivalent to eq. (58). For $k = 2$ we use

$$\frac{\delta^2 \log Z[J]}{\delta J(x)\,\delta J(y)} \;=\; \frac{1}{Z}\,\frac{\delta^2 Z}{\delta J(x)\,\delta J(y)} \;-\; \frac{1}{Z}\,\frac{\delta Z}{\delta J(x)} \times \frac{1}{Z}\,\frac{\delta Z}{\delta J(y)}\,. \tag{61}$$

Let's multiply this formula by $(-i)^2$, set $J \equiv 0$ after taking the derivatives, and apply eqs. (58) to the right hand side; then

$$(-i)^2 \frac{\delta^2 \log Z[J]}{\delta J(x)\,\delta J(y)}\bigg|_{J\equiv 0} \;=\; G_2(x,y) \;-\; G_1(x)\times G_1(y) \;=\; G_2^{\rm conn}(x,y). \tag{62}$$

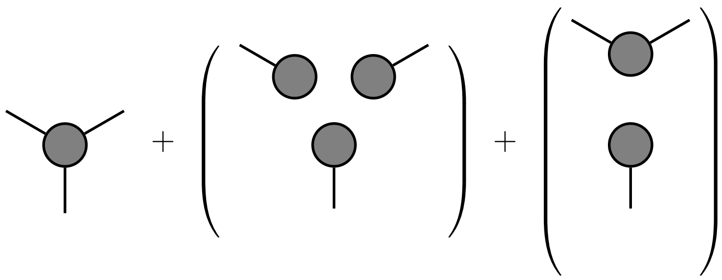
The $k = 1$ and $k = 2$ cases serve as a base of a proof by induction in k. Now we need to prove that is eqs. (60) hold true for all $\ell < k$ that they also hold true for k itself.

A k-point correlation function has a cluster expansion in terms of connected correlation functions,

$$G_k(x_1,\ldots,x_k) \;=\; G_k^{\rm conn}(x_1,\ldots,x_k) \;+\; \sum_{\substack{\text{cluster}\\ \text{decompositions}}} \left(\prod_{\text{clusters}} G^{\rm conn}(\text{cluster})\right) \tag{63}$$

where each cluster is proper subset of $(x_1,\ldots,x_n)$ and we sum over decompositions of the whole set $(x_1,\ldots,x_n)$ into a union of disjoint clusters. For example, for $k = 3$

$$\begin{aligned} G_3(x,y,z) \;=\;& G_3^{\rm conn}(x,y,z) \;+\; G_1(x)\times G_1(y)\times G_1(z)\\ &+\; G_2^{\rm conn}(x,y)\times G_1(z) \;+\; G_2^{\rm conn}(x,z)\times G_1(y)\\ &+\; G_2^{\rm conn}(y,z)\times G_1(x) \end{aligned}$$

=  + two similar. (64)

There is a similar expansion of derivatives of Z in terms of derivatives of $\log Z$,

$$\frac{1}{Z}\,\frac{\delta^k Z}{\delta J(x_1)\cdots\delta J(x_k)} \;=\; \frac{\delta^k \log Z}{\delta J(x_1)\cdots\delta J(x_k)} \;+\; \sum_{\substack{\text{cluster}\\ \text{decompositions}}} \left(\prod_{(x_{(1)},\ldots,x_{(\ell)})}^{\text{clusters}} \frac{\delta^\ell \log Z}{\delta J(x_{(1)})\cdots\delta J(x_{(\ell)})}\right), \tag{65}$$

Now let's set $J(y) \equiv 0$ after taking all the derivatives in this expansion. On the left hand side, this gives us $i^k G_k(x_1, \ldots, x_k)$ by eq. (58). On the right hand side, for each cluster

$$\frac{\delta^\ell \log Z}{\delta J(x_{(1)}) \cdots \delta J(x_{(\ell)})} \;=\; i^\ell G_\ell^{\text{conn}}(x_{(1)}, \ldots, x_{(\ell)}) \tag{66}$$

by the induction assumption. Altogether, we have

$$G_n(x_1, \ldots, x_n) \;=\; \frac{(-i)^k \delta^k \log Z}{\delta J(x_1) \cdots \delta J(x_k)}\bigg|_{J\equiv 0} \;+\; \sum_{\substack{\text{cluster}\\ \text{decompositions}}} \left(\prod_{\text{cluster}} G^{\text{conn}}(\text{cluster}) \right), \tag{67}$$

and comparing this formula to eq. (63) we immediately see that

$$G_n^{\text{conn}}(x_1, \ldots, x_n) \;=\; \frac{(-i)^k \delta^k \log Z}{\delta J(x_1) \cdots \delta J(x_k)}\bigg|_{J\equiv 0}, \tag{60}$$

quod erat demonstrandum.

We may summarize all the equations (60) in a single formula which makes manifest the role of $\log Z$ as a *generating functional* for the connected correlation functions,

$$\log Z[J(x)] \;=\; \log Z[0] \;+\; \sum_{k=1}^{\infty} \frac{i^k}{k!} \int d^4x_1 \cdots \int d^4x_k\, G_k^{\text{conn}}(x_1, \ldots, x_k) \times J(x_1) \cdots J(x_k). \tag{68}$$

To see how this works, consider the free theory as an example. A free scalar field has only one connected correlation function, namely $G_2^{\text{conn}}(x-y) = G_F(x-y)$; all the other G_k^{conn} require interactions and vanish for the free field. Hence, the partition function of the free field should have form

$$\log Z^{\text{free}}[J(x)] \;=\; \log Z^{\text{free}}[0] \;+\; \frac{i^2}{2} \int d^4x \int d^4y\, G_F(x-y) \times J(x) J(y) \;+\; \text{nothing else}. \tag{69}$$

To check this formula, let's calculate the partition function. For the free field Φ,

$$\mathcal{L}^{\text{free}} \;+\; J\Phi \;=\; -\tfrac{1}{2}\Phi(\partial^2 + m^2)\Phi \;+ J\Phi = \;-\tfrac{1}{2}\Phi'(\partial^2 + m^2)\Phi' \;+\; \tfrac{1}{2} J(\partial^2 + m^2)^{-1} J \tag{70}$$

(up to a total derivative) where $\Phi' = \Phi - (\partial^2 + m^2)^{-1} J$. In other words,

$$S^{\rm free}[\Phi(x), J(x)] \stackrel{\rm def}{=} \int (\mathcal{L} + J\Phi) d^4x \;=\; S^{\rm free}[\Phi'(x), \cancel{J}] + \frac{i}{2} \int d^4x \int d^4y \, J(x) G_F(x-y) J(y) \quad (71)$$

where

$$\Phi'(x) \;=\; \Phi(x) \;-\; i \int d^4y \, G_F(x-y) J(y). \quad (72)$$

In the path integral, $\mathcal{D}[\Phi(x)] = \mathcal{D}[\Phi'(x)]$, hence

$$\begin{aligned} Z^{\rm free}[J(x)] &= \iint \mathcal{D}[\Phi(x)] \, \exp\Big(iS^{\rm free}[\Phi(x), J(x)] \Big) \\ &= \iint \mathcal{D}[\Phi'(x)] \, \exp\Big(iS^{\rm free}[\Phi'(x), \cancel{J}] \Big) \times \exp\left(-\frac{1}{2} \int d^4x \int d^4y \, J(x) G_F(x-y) J(y) \right) \\ &= \exp\left(-\frac{1}{2} \int d^4x \int d^4y \, J(x) G_F(x-y) J(y) \right) \times \iint \mathcal{D}[\Phi'(x)] \, \exp\Big(iS^{\rm free}[\Phi'(x), \cancel{J}] \Big) \\ &= \exp\left(-\frac{1}{2} \int d^4x \int d^4y \, J(x) G_F(x-y) J(y) \right) \times Z^{\rm free}[0]. \end{aligned} \quad (73)$$

Taking the log of both sides of this formula gives us

$$\log Z^{\rm free}[J] \;=\; \log Z^{\rm free}[0] \;-\; \frac{1}{2} \int d^4x \int d^4y \, J(x) G_F(x-y) J(y), \quad (74)$$

exactly as in eq. (69), which confirms that $\log Z^{\rm free}[J]$ is indeed the generating functional for the correlation functions of the free field.