1. In class I have explained a free relativistic scalar field into plane wave solutions (of the classical equation of motion) multiplied by annihilation or creation operators. In this problem, you will derive a similar expansion for the massive vector field,

$$\hat{A}_\mu(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \sum_\lambda \left(e^{-ikx} f_\mu(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda}(0) + e^{+ikx} f^*_\mu(\mathbf{k},\lambda)\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_\mathbf{k}} \tag{1}$$

The massive vector field $\hat{A}^\mu(x)$ should be familiar to you from two previous homework sets: in set#1 (problem 1) you've derived its equation of motion from the Lagrangian, while in set#2 (problems 2–3) you've developed the Hamiltonian formalism and quantized the field. For the present exercise you will need the equal-times commutation relations of the quantum fields,

$$[\hat{A}^i(\mathbf{x}),\hat{A}^j(\mathbf{y})] = 0,\quad [\hat{E}^i(\mathbf{x}),\hat{E}^j(\mathbf{y})] = 0,\quad [\hat{A}^i(\mathbf{x}),\hat{E}^j(\mathbf{y})] = -i\delta^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{2}$$

(in $\hbar=1, c=1$ units), the Hamiltonian operator

$$\hat{H} = \int d^3\mathbf{x}\left(\tfrac{1}{2}\hat{\mathbf{E}}^2 + \frac{(\nabla\cdot\hat{\mathbf{E}})^2}{2m^2} + \tfrac{1}{2}(\nabla\times\hat{\mathbf{A}})^2 + \tfrac{1}{2}m^2\hat{\mathbf{A}}^2\right). \tag{3}$$

for the free fields (*i.e.*, for $\hat{J}^\mu(x)\equiv 0$), and the operatorial identity

$$\hat{A}^0(x) = -\frac{1}{m^2}\nabla\cdot\hat{\mathbf{E}}(x) \tag{4}$$

(again, for $\hat{J}^0(x)\equiv 0$).

In general, a QFT has a creation operator $\hat{a}^\dagger_{\mathbf{k},\lambda}$ and an annihilation operator $\hat{a}_{\mathbf{k},\lambda}$ for each plane wave with momentum $\mathbf{k}$ and polarization λ. The massive vector fields have 3 independent polarizations corresponding to 3 orthogonal unit 3–vectors. One may use any basis of 3 such vectors $\mathbf{e}_\lambda(\mathbf{k})$, and it's often convenient to make them $\mathbf{k}$–dependent and complex; in the complex case, orthogonality+unit length mean

$$\mathbf{e}_\lambda(\mathbf{k}) \cdot \mathbf{e}^*_{\lambda'}(\mathbf{k}) \;=\; \delta_{\lambda,\lambda'}\,. \tag{5}$$

Of particular convenience is the helicity basis of eigenvectors of the vector product $i\mathbf{k}\times$, namely

$$i\mathbf{k} \times \mathbf{e}_\lambda(\mathbf{k}) \;=\; \lambda|\mathbf{k}|\mathbf{e}_\lambda(\mathbf{k}), \qquad \lambda = -1, 0, +1. \tag{6}$$

By convention, the phases of the complex helicity eigenvectors are chosen such that

$$\mathbf{e}_0(\mathbf{k}) \;=\; \frac{\mathbf{k}}{|\mathbf{k}|}\,, \quad \mathbf{e}^*_{\pm 1}(\mathbf{k}) \;=\; -\mathbf{e}_{\mp 1}(\mathbf{k}), \quad \mathbf{e}_\lambda(-\mathbf{k}) \;=\; -\mathbf{e}^*_\lambda(+\mathbf{k}), \tag{7}$$

for example, for $\mathbf{k}$ pointing in the positive z direction

$$\mathbf{e}_{+1}(\mathbf{k}) \;=\; \tfrac{1}{\sqrt{2}}(+1,+i,0), \quad \mathbf{e}_{-1}(\mathbf{k}) \;=\; \tfrac{1}{\sqrt{2}}(-1,+i,0), \quad \mathbf{e}_0(\mathbf{k}) \;=\; (0,0,1). \tag{8}$$

(a) As a first step towards constructing the $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ operators, we Fourier transform the vector fields $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$ and then decompose the vectors $\hat{\mathbf{A}}_\mathbf{k}$ and $\hat{\mathbf{E}}_\mathbf{k}$ into helicity components,

$$\begin{aligned}
\hat{\mathbf{A}}(\mathbf{x}) &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{k}\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{A}_{\mathbf{k},\lambda}\,, \qquad & \hat{A}_{\mathbf{k},\lambda} &= \int\! d^3\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\cdot\hat{\mathbf{A}}(\mathbf{x}),\\
\hat{\mathbf{E}}(\mathbf{x}) &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{k}\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{E}_{\mathbf{k},\lambda}\,, \qquad & \hat{E}_{\mathbf{k},\lambda} &= \int\! d^3\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\cdot\hat{\mathbf{E}}(\mathbf{x}).
\end{aligned} \tag{9}$$

Show that $\hat{A}^\dagger_{\mathbf{k},\lambda} = -\hat{A}_{-\mathbf{k},\lambda}$, $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$, and derive the equal-time commutation relations for the $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$ operators.

(b) Show that

$$\hat{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda \left(\frac{C_{\mathbf{k},\lambda}}{2} \hat{E}^\dagger_{\mathbf{k},\lambda} \hat{E}_{\mathbf{k},\lambda} + \frac{\omega^2_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}} \hat{A}^\dagger_{\mathbf{k},\lambda} \hat{A}_{\mathbf{k},\lambda} \right)$$

$$\text{where} \quad \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}, \tag{10}$$

$$\text{and} \quad C_{\mathbf{k},\lambda} = \begin{cases} \omega^2_{\mathbf{k}}/m^2 & \text{for } \lambda = 0, \\ 1 & \text{for } \lambda = \pm 1. \end{cases}$$

(c) Define the annihilation and the creation operators according to

$$\hat{a}_{\mathbf{k},\lambda} = \frac{\omega_{\mathbf{k}} \hat{A}_{\mathbf{k},\lambda} - iC_{\mathbf{k},\lambda} \hat{E}_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}}, \qquad \hat{a}^\dagger_{\mathbf{k},\lambda} = \frac{\omega_{\mathbf{k}} \hat{A}^\dagger_{\mathbf{k},\lambda} + iC_{\mathbf{k},\lambda} \hat{E}^\dagger_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}}, \tag{11}$$

and verify that they satisfy the relativistically normalized bosonic commutation relations. (At equal times or in the Schrödinger picture.)

(d) Show that

$$\hat{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2\omega_{\mathbf{k}}} \sum_\lambda \omega_{\mathbf{k}}\, \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \text{const.} \tag{12}$$

Note: by a constant here I mean a c-number rather than an operator. It happens to be a badly divergent number, but that's OK.

(e) Next, consider the time dependence of the free vector field in the Heisenberg picture. Show that

$$\hat{\mathbf{A}}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2\omega_{\mathbf{k}}} \sum_\lambda \sqrt{C_{\mathbf{k},\lambda}} \left(e^{-ikx} \mathbf{e}_\lambda(\mathbf{k})\, \hat{a}_{\mathbf{k},\lambda}(0) + e^{+ikx} \mathbf{e}^*_\lambda(\mathbf{k})\, \hat{a}^\dagger_{\mathbf{k},\lambda}(0) \right)_{k^0 = +\omega_{\mathbf{k}}}. \tag{13}$$

(f) Write down similar expansion for the electric field $\hat{\mathbf{E}}(\mathbf{x},t)$ and the scalar potential $\hat{A}^0(\mathbf{x},t)$; use eq. (4) for the latter.

(g) Combine the results of parts (e) and (f) into a relativistic formula (1) for the 4–vector field $\hat{A}^\mu(x)$. The polarization 4–vectors $f^\mu(\mathbf{k},\lambda)$ in that formula should be

$$f^\mu(\mathbf{k},\lambda) \;=\; \begin{cases} \bigl(0,\mathbf{e}_\lambda(\mathbf{k})\bigr) & \text{for } \lambda=\pm 1,\\[1ex] \left(\frac{|\mathbf{k}|}{m},\,\frac{\omega_{\mathbf{k}}}{m}\,\frac{\mathbf{k}}{|\mathbf{k}|}\right) & \text{for } \lambda=0. \end{cases} \tag{14}$$

(h) Show that these polarization 4–vectors obtain by Lorentz boosting of the purely-spatial polarization vectors $(0,\mathbf{e}_\lambda(\mathbf{k}))$ into the frame of the wave moving with the velocity $\mathbf{v}=\mathbf{k}/\omega_k$.

Also, check that the $f^\mu(\mathbf{k},\lambda)$ satisfy

$$k_\mu f^\mu_{\mathbf{k},\lambda} \;=\; 0, \qquad f^\mu_{\mathbf{k},\lambda}\left(f^*_{\mathbf{k},\lambda'}\right)_\mu \;=\; -\delta_{\lambda,\lambda'}\,. \tag{15}$$

(i) Finally, verify that the quantum vector field (1) satisfies the free equations of motion $\partial_\mu\hat{A}^\mu(x)=0$ and $(\partial^2+m^2)\hat{A}^\mu(x)=0$; moreover, each mode in the expansion (1) satisfies the equations of motions without any help from the other modes.

2. The ordinary quantum mechanics of a single relativistic particle — or any fixed number of relativistic particles — violates the relativistic causality by allowing particles to move faster than light. In this problem, we shall see how this works for the simplest case of a single free relativistic spinless particle with the Hamiltonian

$$\hat{H} \;=\; +\sqrt{m^2+\hat{\mathbf{P}}^2} \tag{16}$$

(in the $c=\hbar=1$ units). By general rules of quantum mechanics, the amplitude $U(x\to y)$ for this particle to propagate from point $\mathbf{x}$ at time x^0 to point $\mathbf{y}$ at time y^0 obtains from the Hamiltonian (16) as

$$U(x\to y) \;=\; \bigl\langle \mathbf{y},y^0|\mathbf{x},x^0\bigr\rangle^{\text{Heisenberg}}_{\text{picture}} \;=\; \langle\mathbf{y}|\exp\bigl(-i(y^0-x^0)\hat{H}\bigr)\,|\mathbf{x}\rangle^{\text{Schroedinger}}_{\text{picture}}\,. \tag{17}$$

(a) Use momentum basis for the Hamiltonian (16) to evaluate the coordinate-basis evolution kernel (17) as

$$U(x\to y) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\exp\Bigl(i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})-i\omega(\mathbf{k})\times(y^0-x^0)\Bigr) \tag{18}$$

$$\text{for} \quad \omega(\mathbf{k}) \stackrel{\text{def}}{=} +\sqrt{m^2+\mathbf{k}^2}\,, \tag{19}$$

then reduce the 3D momentum integral to the one-dimensional integral

$$U(x\to y) = \frac{-i}{4\pi^2\, r} \int\limits_{-\infty}^{+\infty} dk\, k\, \exp\bigl(irk - it\omega(k)\bigr) \tag{20}$$

where $r=|\mathbf{y}-\mathbf{x}|$ and $t=y^0-x^0$.

We are particularly interested in the asymptotic behavior of the integral (20) in the limit of $r\to\infty$, $t\to\infty$, fixed t/r ratio. The best method for obtaining the asymptotic behavior of such integrals — or more general integrals of the form

$$\int dx\, f(x)\times\exp\bigl(-Ag(x)\bigr), \qquad A\to\infty \tag{21}$$

is the *saddle-point method* (AKA the mountain-pass method).

(b) If you are not familiar with the saddle-point method, read my notes on it.
Those notes were originally written for a QM class, so they include the Airy function example and the relation of the Airy functions to the WKB approximation. You do not need the WKB or the Airy functions for this homework, just the saddle-point method itself, so focus on the first 6 pages of my notes, the rest is optional.

(c) Now let's use the saddle point method to evaluate the integral (20) in the limit of $r\to\infty$, $t\to\infty$, while the ratio r/t stays fixed. Specifically, let $(r/t)<1$ so we stay inside the future light cone.
Show that in this limit, the evolution kernel (20) becomes

$$U(x\to y) \approx \left(\frac{-iM}{2\pi}\right)^{3/2} \times \frac{t}{(t^2-r^2)^{5/4}} \times \exp(-iM\sqrt{t^2-r^2}). \tag{22}$$

(d) Finally, take a similar limit but go outside the light cone, thus fixed $(r/t)>1$ while $r,t\to+\infty$. Show that in this limit, the kernel becomes

$$U(x\to y) \approx \frac{iM^{3/2}}{(2\pi)^{3/2}} \times \frac{t}{(r^2-t^2)^{5/4}} \times \exp(-M\sqrt{r^2-t^2}). \tag{23}$$

Hint: for $r>t$ the saddle point is at complex k.

Eq. (23) shows that the propagation amplitude $U(x \to y)$ diminishes exponentially outside the light cone, *but it does not vanish!* Thus, given a particle localized at point $\mathbf{x}$ at the time x^0, at a later time $y^0 = x^0 + t$ the wave function is *mostly* limited to the future light cone $r < t$, *but there is an exponential tail outside the light cone.* In other words, the probability of superluminal motion is exponentially small but non-zero.

Obviously, such superluminal propagation cannot be allowed in a consistently relativistic theory. And that's why relativistic quantum mechanics of a single particle is inconsistent. Likewise, relativistic quantum mechanics of any fixed number of particles does not work, except as an approximation.

In the quantum field theory, this paradox is resolved by allowing for creation and annihilation of particles. Quantum field operators acting at points x and y outside each others' future lightcones can either create a particle at x and then annihilate it at y, or else annihilate it at y and then create it at x. I will show in class that the two effects *precisely* cancel each other, so altogether there is no propagation outside the light cone. That's how relativistic QFT is perfectly causal while the relativistic QM is not.