1. Consider once again the massive vector field $\hat{A}^\mu(x)$. In the previous homework (set#3, problem 2), you (should have) expanded the free vector field into creation and annihilation operators multiplied by the plane-waves according to

$$\hat{A}^\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}} \sum_\lambda \Big(e^{-ikx} \times f^\mu_{\mathbf{k},\lambda} \times \hat{a}_{\mathbf{k},\lambda} + e^{+ikx} \times f^{*\mu}_{\mathbf{k},\lambda} \times \hat{a}^\dagger_{\mathbf{k},\lambda} \Big)^{k^0=+\omega_\mathbf{k}}. \tag{1}$$

The λ here labels the independent polarizations of a vector particle (for example, the helicities $\lambda = -1, 0, +1$), while $f^\mu_{\mathbf{k},\lambda}$ are the polarization vectors obeying

$$k_\mu f^\mu_{\mathbf{k},\lambda} = 0, \quad g_{\mu\nu} f^\mu_{\mathbf{k},\lambda} f^{*\nu}_{\mathbf{k},\lambda'} = -\delta_{\lambda,\lambda'}\,. \tag{2}$$

In this problem, we shall calculate the Feynman propagator for the massive vector field (1).

(a) First, a lemma: Show that any polarization vectors obeying the constraints (2) also obey

$$\sum_\lambda f^\mu_{\mathbf{k},\lambda} f^{*\nu}_{\mathbf{k},\lambda} = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\,. \tag{3}$$

(b) Next, calculate the "vacuum sandwich" of two vector fields and show that

$$\begin{aligned} \langle 0|\,\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}} \left[\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) e^{-ik(x-y)} \right]_{k^0=+\omega_\mathbf{k}} \\ &= \left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2} \right) D(x-y). \end{aligned} \tag{4}$$

(c) Now consider a free scalar field (of the same mass m as the vector field) and its Feynman propagator $G_F^{\rm scalar}(x-y)$. Show that

$$\left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2} \right) G_F^{\rm scalar}(x-y) = \langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle + \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{5}$$

To avoid the δ–function singularity in formulae like (5), the time-ordered product of the vector fields (or rather, just of their $\hat{A}^0$ components) is *modified** according to

$$\mathbf{T}^* \hat{A}^\mu(x)\hat{A}^\nu(y) \;=\; \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y) \;+\; \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{6}$$

Consequently, the Feynman propagator for the massive vector field is defined using the modified time-ordered product of the two fields,

$$G_F^{\mu\nu}(x-y) \;\stackrel{\text{def}}{=}\; \langle 0|\,\mathbf{T}^* \hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \tag{7}$$

(d) Show that this propagator obtains as

$$G_F^{\mu\nu}(x-y) \;=\; \int\!\frac{d^4\mathbf{k}}{(2\pi)^4}\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)\times\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\,. \tag{8}$$

(e) Finally, write the classical action for the free vector field as

$$S \;=\; \tfrac{1}{2}\int\! d^4x\, A_\mu(x)\,\mathcal{D}^{\mu\nu}A_\nu(x) \tag{9}$$

where $\mathcal{D}^{\mu\nu}$ is a differential operator, and show that the Feynman propagator (8) is a Green's function of this operator,

$$\mathcal{D}_x^{\mu\nu}G^F_{\nu\lambda}(x-y) \;=\; +i\delta^\mu_\lambda\delta^{(4)}(x-y). \tag{10}$$

2. Next, a reading assignment. To help you understand the relations between the continuous symmetries, their generators, the multiplets, and the representations of the generators and of the finite symmetries, read about the rotational symmetry and its generators in chapter 3 of the J. J. Sakurai's book *Modern Quantum Mechanics*.† Please focus on sections 1, 2, 3, second half of section 5 (representations of the rotation operators), and section 10; the other sections 4, 6, 7, 8, and 9 are not relevant to the present class material.

   PS: If you have already read the Sakurai's book before but it has been a while, please read it again.

$\star$ See *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.

$\dagger$ The UT Math–Physics–Astronomy library has several hard copies but no electronic copies of the book. However, you can find several pirate scans of the book (in PDF format) all over the web; Google them up if you cannot find a legitimate copy.

3. Finally consider N interacting real scalar fields $\Phi_1, \ldots, \Phi_N$ with an $O(N)$ symmetric Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2}\sum_{a=1}^{N}\bigl(\partial_\mu\Phi_a\bigr)^2 \;-\; \frac{m^2}{2}\sum_{a=1}^{N}\Phi_a^2 \;-\; \frac{\lambda}{24}\left(\sum_{a=1}^{N}\Phi_a^2\right)^2 . \tag{11}$$

By the Noether theorem, the continuous $SO(N)$ subgroup of the $O(N)$ symmetry gives rise to $\frac{1}{2}N(N-1)$ conserved currents

$$J^\mu_{ab}(x) \;=\; -J^\mu_{ba}(x) = \;\Phi_a(x)\,\partial^\mu\Phi_b(x) -\; \Phi_b(x)\,\partial^\mu\Phi_a(x). \tag{12}$$

In the quantum field theory, these currents become operators

$$\begin{aligned}\hat{\mathbf{J}}_{ab}(\mathbf{x},t) \;&=\; -\hat{\mathbf{J}}_{ba}(\mathbf{x},t) \;=\; -\hat\Phi_a(\mathbf{x},t)\nabla\hat\Phi_b(\mathbf{x},t) \;+\; \hat\Phi_b(\mathbf{x},t)\nabla\hat\Phi_a(\mathbf{x},t),\\ \hat J^0_{ab}(\mathbf{x},t) \;&=\; -\hat J^0_{ba}(\mathbf{x},t) \;=\; \hat\Phi_a(\mathbf{x},t)\hat\Pi_b(\mathbf{x},t) \;-\; \hat\Phi_b(\mathbf{x},t)\hat\Pi_a(\mathbf{x},t).\end{aligned} \tag{13}$$

This problem is about the net charge operators

$$\hat Q_{ab}(t) \;=\; -\hat Q_{ba}(t) \;=\int d^3\mathbf{x}\,\hat J^0_{ab}(\mathbf{x},t) \;=\int d^3\mathbf{x}\left(\hat\Phi_a(\mathbf{x},t)\hat\Pi_b(\mathbf{x},t) \;-\; \hat\Phi_b(\mathbf{x},t)\hat\Pi_a(\mathbf{x},t)\right). \tag{14}$$

(a) Write down the equal-time commutation relations for the quantum $\hat\Phi_a$ and $\hat\Pi_a$ fields. Also, write down the Hamiltonian operator for the interacting fields.

(b) Show that

$$\begin{aligned}\Bigl[\hat Q_{ab}(t), \hat\Phi_c(\mathbf{x}, \text{same } t)\Bigr] \;&=\; -i\delta_{bc}\hat\Phi_a(\mathbf{x},t) \;+\; i\delta_{ac}\hat\Phi_b(\mathbf{x},t),\\ \Bigl[\hat Q_{ab}(t), \hat\Pi_c(\mathbf{x}, \text{same } t)\Bigr] \;&=\; -i\delta_{bc}\hat\Pi_a(\mathbf{x},t) \;+\; i\delta_{ac}\hat\Pi_b(\mathbf{x},t),\end{aligned} \tag{15}$$

(c) Show that the all the $\hat Q_{ab}$ commute with the Hamiltonian operator $\hat H$. In the Heisenberg picture, this makes all the charge operators $\hat Q_{ab}$ time independent.

(d) Verify that the $\hat Q_{ab}$ obey commutation relations of the $SO(N)$ generators,

$$\Bigl[\hat Q_{ab}, \hat Q_{cd}\Bigr] \;=\; -i\delta_{[c[b}\hat Q_{a]d]} \;\equiv\; -i\delta_{bc}\hat Q_{ad} \;+\; i\delta_{ac}\hat Q_{bd} \;+\; i\delta_{bd}\hat Q_{ac} \;-\; i\delta_{ad}\hat Q_{bc}\,. \tag{16}$$

Now let's take $\lambda \to 0$ and focus on the free fields. Let's work in the Schrödinger picture and expand all the fields into creation and annihilation operators $\hat a^\dagger_{\mathbf{p},a}$ and $\hat a_{\mathbf{p},a}$ $(a = 1, \ldots, N)$.

(e) Show that in terms of creation and annihilation operators, the charges (14) become

$$\hat{Q}_{ab} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \left(-i\hat{a}^\dagger_{\mathbf{p},a}\hat{a}_{\mathbf{p},b} + i\hat{a}^\dagger_{\mathbf{p},b}\hat{a}_{\mathbf{p},a} \right). \tag{17}$$

For $N = 2$, the $SO(2)$ symmetry becomes the $U(1)$ phase symmetry one complex field $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and its conjugate $\Phi^* = (\Phi_1 - i\Phi_2)/\sqrt{2}$,

$$\Phi(x) \rightarrow e^{-i\theta}\Phi(x), \quad \Phi^*(x) \rightarrow e^{+i\theta}\Phi^*(x). \tag{18}$$

In the Fock space, the corresponding quantum fields $\hat{\Phi}(x)$ and $\hat{\Phi}^\dagger(x)$ give rise to particles and anti-particles of opposite charges; the creation and annihilation operators for such particles and antiparticles are

$$\begin{aligned}
\hat{a}_\mathbf{p} &= \frac{\hat{a}_{\mathbf{p},1} + i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} \quad \text{are particle annihilation operators,} \\
\hat{b}_\mathbf{p} &= \frac{\hat{a}_{\mathbf{p},1} - i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} \quad \text{are antiparticle annihilation operators,} \\
\hat{a}^\dagger_\mathbf{p} &= \frac{\hat{a}^\dagger_{\mathbf{p},1} - i\hat{a}^\dagger_{\mathbf{p},2}}{\sqrt{2}} \quad \text{are particle creation operators,} \\
\hat{b}^\dagger_\mathbf{p} &= \frac{\hat{a}^\dagger_{\mathbf{p},1} + i\hat{a}^\dagger_{\mathbf{p},2}}{\sqrt{2}} \quad \text{are antiparticle creation operators.}
\end{aligned} \tag{19}$$

(f) Show that in terms of the operators (19),

$$\hat{Q}_{21} = -\hat{Q}_{12} = \hat{N}_{\text{particles}} - \hat{N}_{\text{antiparticles}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \left(\hat{a}^\dagger_\mathbf{p}\hat{a}_\mathbf{p} - \hat{b}^\dagger_\mathbf{p}\hat{b}_\mathbf{p} \right). \tag{20}$$

(g) In terms of $\hat{\Phi}$ and $\hat{\Phi}^\dagger$, the commutation relations (15) become

$$[\hat{Q}_{21}, \hat{\Phi}(x)] = -\hat{\Phi}(x), \quad [\hat{Q}_{21}, \hat{\Phi}^\dagger(x)] = +\hat{\Phi}^\dagger(x). \tag{21}$$

Verify these commutators, then use the Hadamard Lemma

$$\begin{aligned}
e^{\hat{A}}\hat{B}e^{-\hat{A}} &= \sum_{n=1}^{\infty} [\hat{A}, \ldots, [\hat{A}, \hat{B}]\cdots]_{n\,\text{times}} \\
&= B + [\hat{A}, \hat{B}] + \tfrac{1}{2}[\hat{A}, [\hat{A}, \hat{B}]] + \tfrac{1}{6}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \cdots
\end{aligned} \tag{22}$$

to show that the charge $\hat{Q}_{21}$ *generates* the phase symmetry (18) according to

$$\begin{aligned} \exp(+i\theta\hat{Q}_{21})\hat{\Phi}(x)\exp(-i\theta\hat{Q}_{21}) &= e^{-i\theta}\hat{\Phi}(x), \\ \exp(+i\theta\hat{Q}_{21})\hat{\Phi}^\dagger(x)\exp(-i\theta\hat{Q}_{21}) &= e^{+i\theta}\hat{\Phi}^\dagger(x). \end{aligned} \tag{23}$$

Now let's go back to $N > 2$ and show that the charges $\hat{Q}_{ab}$ generate the $SO(N)$ symmetry of the quantum fields. Any $SO(N)$ rotation matrix R can be written as a matrix exponential of an antisymmetric matrix, $R = \exp(A)$ for $A^\top = -A$. For this matrix A, let's define a unitary operator in the Fock space

$$\hat{U} = \exp\left(-\frac{i}{2}\sum_{ab} A_{ab}\hat{Q}_{ab}\right). \tag{24}$$

(h) Verify that this operator is indeed unitary for any real antisymmetric matrix A.

(i) Show that $\hat{U}$ implements the $SO(N)$ rotation R in the scalar field space,

$$\hat{U}\hat{\Phi}_a(x)\hat{U}^\dagger = \sum_b R_{ab}\hat{\Phi}_b\,. \tag{25}$$

Hint: use the commutation relations (15) and the Hadamard lemma (22).

(j) Argue that $[\hat{Q}_{ab}, \hat{H}] = 0$ and eq. (25) for the action of the $\hat{U}$ symmetries on the quantum fields together imply simlar transformation laws for the creation and the annihilation operators

$$\hat{U}\hat{a}_{\mathbf{p},a}\hat{U}^\dagger = \sum_b R_{ab}\hat{a}_{\mathbf{p},b} \quad \text{and} \quad \hat{U}\hat{a}^\dagger_{\mathbf{p},a}\hat{U}^\dagger = \sum_b R_{ab}\hat{a}^\dagger_{\mathbf{p},b}\,. \tag{26}$$

(k) Finally, show that when $\hat{U}$ acts on a multiparticle state, it rotates the species index of each particle by R,

$$\hat{U}\,|n:(\mathbf{p}_1,a_1),\ldots,(\mathbf{p}_n,a_n)\rangle = \sum_{b_1,\ldots,b_n} R_{a_1,b_1}\cdots R_{a_n,b_n}\,|n:(\mathbf{p}_1,b_1),\ldots,(\mathbf{p}_n,b_n)\rangle\,. \tag{27}$$

Note: for simplicity assume that all particles have different momenta, $\mathbf{p}_1 \neq \mathbf{p}_2$, *etc.*, then use part (j).