This homework has 4 problems. Problems 1 and 2 are about the stress-energy tensor for the EM fields: free EM fields in problem 1, and EM fields coupled to charged scalr fields in problem 2. The other two problems 3 and 4 are about non-abelian gauge theories. Altogether, it's a pretty large homework set, so start working early.

1. According to the Noether theorem, a translationally invariant system of classical fields $\phi_a(x)$ has a conserved stress-energy tensor

$$T^{\mu\nu}_{\text{Noether}} \;=\sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi_a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{1}$$

For the scalar fields, real or complex, this Noether stress-energy tensor is is properly symmetric, $T^{\mu\nu}_{\text{Noether}} = T^{\nu\mu}_{\text{Noether}}$. But for the vector, tensor, spinor, *etc.*, fields, the Noether stress-energy tensor (1) comes out asymmetric, so to make it properly symmetric one adds a total-divergence term of the form

$$T^{\mu\nu} \;=\; T^{\mu\nu}_{\text{Noether}} \;+\; \partial_\lambda\mathcal{K}^{\lambda\mu\,\nu}, \tag{2}$$

where $\mathcal{K}^{\lambda\mu\,\nu} \equiv -\mathcal{K}^{\mu\lambda\,\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices.

To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu,\partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \tag{3}$$

where A_μ is a real vector field and $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Write down $T^{\mu\nu}_{\text{Noether}}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(b) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\rm EM} \;=\; -F^{\mu\lambda}F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}\,g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \tag{4}$$

Show that this expression indeed has form (2) for

$$\mathcal{K}^{\lambda\mu,\nu} \;=\; -F^{\lambda\mu}A^{\nu} \;=\; -\mathcal{K}^{\mu\lambda,\nu}. \tag{5}$$

(c) Write down the components of the stress-energy tensor (4) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density, and stress.

Next, consider the electromagnetic fields coupled to the electric current J^{μ} of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate $P^{\mu}_{\rm EM}$ and $P^{\mu}_{\rm mat}$. Consequently, we should have

$$\partial_{\mu}T^{\mu\nu}_{\rm net} \;=\; 0 \qquad \text{for} \quad T^{\mu\nu}_{\rm net} \;=\; T^{\mu\nu}_{\rm EM} \;+\; T^{\mu\nu}_{\rm mat} \tag{6}$$

but generally $\partial_{\mu}T^{\mu\nu}_{\rm EM} \neq 0$ and $\partial_{\mu}T^{\mu\nu}_{\rm mat} \neq 0$.

(d) Use Maxwell's equations to show that

$$\partial_{\mu}T^{\mu\nu}_{\rm EM} \;=\; -F^{\nu\lambda}J_{\lambda} \tag{7}$$

(in $c = 1$ units), and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_{λ} according to

$$\partial_{\mu}T^{\mu\nu}_{\rm mat} \;=\; +F^{\nu\lambda}J_{\lambda}. \tag{8}$$

(e) Rewrite eq. (7) in non-relativistic notations and explain its physical meaning in terms of the electromagnetic energy, momentum, work, and forces.

2. Continuing problem 1, consider the EM fields coupled to a specific model of charged matter, namely a complex scalar field $\Phi(x) \neq \Phi^*(x)$ of electric charge $q \neq 0$. Altogether, the net Lagrangian for the A^μ, Φ, and Φ^* fields is

$$\mathcal{L}_{\text{net}} = D^\mu\Phi^* D_\mu\Phi - m^2\Phi^*\Phi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{9}$$

where

$$D_\mu\Phi = (\partial_\mu + iqA_\mu)\Phi \quad \text{and} \quad D_\mu\Phi^* = (\partial_\mu - iqA_\mu)\Phi^* \tag{10}$$

are the *covariant* derivatives.

(a) Write down the equation of motion for all fields in a covariant from. Also, write down the electric current

$$J^\mu \stackrel{\text{def}}{=} -\frac{\partial\mathcal{L}}{\partial A_\mu} \tag{11}$$

in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

(b) Write down the Noether stress-energy tensor for the whole system and show that

$$T^{\mu\nu}_{\text{net}} \equiv T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{mat}} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda\mathcal{K}^{\lambda\mu\nu}, \tag{12}$$

where $T^{\mu\nu}_{\text{EM}}$ is exactly as in eq. (4) for the free EM fields, the improvement tensor $\mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu}$ is also exactly as in eq. (5), and

$$T^{\mu\nu}_{\text{mat}} = D^\mu\Phi^*\, D^\nu\Phi + D^\nu\Phi^*\, D^\mu\Phi - g^{\mu\nu}\big(D_\lambda\Phi^*\, D^\lambda\Phi - m^2\Phi^*\,\Phi\big). \tag{13}$$

Note: although the improvement tensor $\mathcal{K}^{\lambda\mu\nu}$ for the EM + matter system is the same as for the free EM fields, in presence of an electric current J^μ its derivative $\partial_\lambda\mathcal{K}^{\lambda\mu\nu}$ contains an extra $J^\mu A^\nu$ term. Pay attention to this term — it is important for obtaining the gauge-invariant stress-energy tensor (13) for the scalar field.

(c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_\mu, D_\nu]\Phi \;=\; iqF_{\mu\nu}\Phi, \qquad [D_\mu, D_\nu]\Phi^* \;=\; -iqF_{\mu\nu}\Phi^* \tag{14}$$

to show that

$$\partial_\mu T^{\mu\nu}_{\text{mat}} \;=\; +F^{\nu\lambda}J_\lambda \tag{15}$$

exactly as in eq. (8), and therefore the *net* stress-energy tensor (12) is conserved, *cf.* problem **1**(d).

3. In class, I have focused on the *fundamental multiplet* of the local $SU(N)$ symmetry, *i.e.*, a set of N fields (complex scalars or Dirac fermions) which transform as a complex N–vector,

$$\Psi'(x) \;=\; U(x)\Psi(x) \quad i.e. \quad \Psi'_i(x) \;=\sum_j U_i^{\,j}(x)\Psi_j(x), \quad i,j=1,2,\ldots,N \tag{16}$$

where $U(x)$ is an x–dependent unitary $N\times N$ matrix, $\det U(x)\equiv 1$. Now consider N^2-1 real fields $\Phi^a(x)$ forming an *adjoint multiplet*: In matrix form

$$\Phi(x) \;=\; \sum_a \Phi^a(x)\times\frac{\lambda^a}{2} \tag{17}$$

is a traceless hermitian $N\times N$ matrix which transforms under the local $SU(N)$ symmetry as

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x). \tag{18}$$

Note that this transformation law preserves the $\Phi^\dagger=\Phi$ and the $\mathrm{tr}(\Phi)=0$ conditions. The covariant derivatives D_μ act on an adjoint multiplet according to

$$D_\mu\Phi(x) \;=\; \partial_\mu\Phi(x) + i[\mathcal{A}_\mu(x),\Phi(x)] \;\equiv\; \partial_\mu\Phi(x) + i\mathcal{A}_\mu(x)\Phi(x) - i\Phi(x)\mathcal{A}_\mu(x), \tag{19}$$

or in components

$$D_\mu\Phi^a(x) = \;\partial_\mu\Phi_a(x) \;-\; f^{abc}\mathcal{A}^b_\mu(x)\Phi^c(x). \tag{20}$$

(a) Verify that these derivatives are indeed covariant under finite gauge transforms.

(b) Verify the Leibniz rule for covariant derivatives of matrix products. Let $\Phi(x)$ and $\Xi(x)$ be two adjoint multiplets while $\Psi(x)$ is a fundamental multiplet and $\Psi^\dagger(x)$ is its hermitian conjugate (row vector of Ψ_i^*). Show that

$$\begin{aligned}
D_\mu(\Phi\Xi) &= (D_\mu\Phi)\Xi + \Phi(D_\mu\Xi), \\
D_\mu(\Phi\Psi) &= (D_\mu\Phi)\Psi + \Phi(D_\mu\Psi), \\
D_\mu(\Psi^\dagger\Xi) &= (D_\mu\Psi^\dagger)\Xi + \Psi^\dagger(D_\mu\Xi).
\end{aligned} \tag{21}$$

(c) Show that for an adjoint multiplet $\Phi(x)$,

$$[D_\mu, D_\nu]\Phi(x) = i[\mathcal{F}_{\mu\nu}(x), \Phi(x)] = ig[F_{\mu\nu}(x), \Phi(x)] \tag{22}$$

or in components $[D_\mu, D_\nu]\Phi^a(x) = -gf^{abc}F^b_{\mu\nu}(x)\Phi^c(x)$.

- In my notations A_μ and $F_{\mu\nu}$ are canonically normalized fields while $\mathcal{A}_\mu = gA_\mu$ and $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$ are normalized by the symmetry action.

In class, I have argued (using covariant derivatives) that the tension fields $\mathcal{F}_{\mu\nu}(x)$ themselves transform according to eq. (18). In other words, the $\mathcal{F}^a_{\mu\nu}(x)$ form an adjoint multiplet of the $SU(N)$ symmetry group.

(d) Verify the $\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$ transformation law directly from the definition $\mathcal{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$ and the non-abelian gauge transform of the $\mathcal{A}_\mu$ fields.

(e) Verify the covariant differential identity for the non-abelian tension fields $\mathcal{F}_{\mu\nu}(x)$:

$$D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu} = 0. \tag{23}$$

Note the covariant derivatives (19) in this equation.

Finally, consider the $SU(N)$ Yang–Mills theory — the non-abelian gauge theory that does not have any fields except $\mathcal{A}^a(x)$ and $\mathcal{F}^a(x)$; its Lagrangian is

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2g^2}\,\text{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) = \sum_a \tfrac{-1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{24}$$

(f) Show that the Euler–Lagrange field equations for the Yang–Mills theory can be written in covariant form as $D_\mu\mathcal{F}^{\mu\nu} = 0$.

Hint: first show that for an infinitesimal variation $\delta\mathcal{A}_\mu(x)$ of the non-abelian gauge fields, the tension fields vary according to $\delta\mathcal{F}_{\mu\nu}(x) = D_\mu\delta\mathcal{A}_\nu(x) - D_\nu\delta\mathcal{A}_\mu(x)$.

4. Continuing the previous problem, consider an $SU(N)$ gauge theory in which N^2-1 vector fields $A^a_\mu(x)$ interact with some "matter" fields $\phi_\alpha(x)$,

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;+\; \mathcal{L}_{\mathrm{mat}}(\phi, D_\mu\phi). \tag{25}$$

For the moment, let me keep the matter fields completely generic — they can be scalars, or vectors, or spinors, or whatever, and form any kind of a multiplet of the local $SU(N)$ symmetry as long as such multiplet is complete and non-trivial. All we need to know right now is that there are well-defined covariant derivatives $D_\mu\phi$ that depend on the gauge fields A^a_μ, which give rise to the currents

$$J^{a\mu} \;=\; -\frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial A^a_\mu} \;=\; -\sum_\phi \frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial(D_\mu\phi)} \times ig\hat{T}^a\phi. \tag{26}$$

Collectively, these N^2-1 currents should form an adjoint multiplet $J^\mu = \sum_a(\frac{1}{2}\lambda^a)J^{a\mu}$ of the $SU(N)$ symmetry.

(a) Show that in this theory the equation of motion for the A^a_μ fields are $D_\mu F^{a\mu\nu} = J^{a\nu}$ and that consistency of these equations requires require the currents to be *covariantly conserved,*

$$D_\mu J^\mu \;=\; \partial_\mu J^\mu \;+\; i[\mathcal{A}_\mu, J^\mu] \;=\; 0, \tag{27}$$

or in components, $\partial_\mu J^{a\mu} - f^{abc}\mathcal{A}^b_\mu J^{c\mu} = 0$.

Note: a covariantly conserved current does *not* lead to a conserved charge, $(d/dt)\int d^3\mathbf{x}\, J^{a0}(\mathbf{x},t) \neq 0$!

Now consider a simple example of matter fields — a fundamental multiplet $\Psi(x)$ of N scalar fields $\Psi^i(x)$, with a Lagrangian

$$\mathcal{L}_{\mathrm{mat}} \;=\; D_\mu\Psi^\dagger D^\mu\Psi \;-\; m^2\Psi^\dagger\Psi \;-\; \frac{\lambda}{4}(\Psi^\dagger\Psi)^2, \qquad \mathcal{L}_{\mathrm{net}} \;=\; \mathcal{L}_{\mathrm{mat}} \;-\; \frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big). \tag{28}$$

(b) Derive the $SU(N)$ currents $J^{a\mu}$ for this set of fields and verify that under $SU(N)$

symmetries the currents transform covariantly into each other as members of the adjoint multiplet. That is, the $N \times N$ matrix $J^\mu = \sum_a (\frac{1}{2}\lambda^a) J^{a\mu}$ transforms according to eq. (18).

Hint: for any complex vectors Ψ and Ψ', $\sum_a (\Psi^\dagger \lambda^a \Psi')\lambda^a = 2\Psi' \otimes \Psi^\dagger - \frac{2}{N}(\Psi^\dagger \Psi') \times \mathbf{1}$.

(c) Finally, verify the covariant conservation $D_\mu J^{a\mu}$ of these currents when the scalar fields $\Psi^i(x)$ and $\Psi^\dagger_i(x)$ obey their equations of motion.