This homework has 4 problems. Problems 1 and 2 are about the stress-energy tensor for the EM fields: free EM fields in problem 1, and EM fields coupled to charged scalr fields in problem 2. The other two problems 3 and 4 are about non-abelian gauge theories. Altogether, it's a pretty large homework set, so start working early.

1. According to the Noether theorem, a translationally invariant system of classical fields $\phi_a(x)$ has a conserved stress-energy tensor

$$T_{\text{Noether}}^{\mu\nu} = \sum_{a} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi_{a})} \partial^{\nu}\phi_{a} - g^{\mu\nu}\mathcal{L}.$$
 (1)

For the scalar fields, real or complex, this Noether stress-energy tensor is is properly symmetric, $T_{\text{Noether}}^{\mu\nu} = T_{\text{Noether}}^{\nu\mu}$. But for the vector, tensor, spinor, *etc.*, fields, the Noether stress-energy tensor (1) comes out asymmetric, so to make it properly symmetric one adds a total-divergence term of the form

$$T^{\mu\nu} = T^{\mu\nu}_{\text{Noether}} + \partial_{\lambda} \mathcal{K}^{\lambda\mu\nu}, \qquad (2)$$

where $\mathcal{K}^{\lambda\mu\nu} \equiv -\mathcal{K}^{\mu\lambda\nu}$ is some 3-index Lorentz tensor antisymmetric in its first two indices.

To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_{\mu}, \partial_{\nu}A_{\mu}) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$
(3)

where A_{μ} is a real vector field and $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

(a) Write down $T_{\text{Noether}}^{\mu\nu}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(b) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T_{\rm EM}^{\mu\nu} = -F^{\mu\lambda}F^{\nu}_{\ \lambda} + \frac{1}{4}g^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}.$$
(4)

Show that this expression indeed has form (2) for

$$\mathcal{K}^{\lambda\mu,\nu} = -F^{\lambda\mu}A^{\nu} = -\mathcal{K}^{\mu\lambda,\nu}.$$
 (5)

(c) Write down the components of the stress-energy tensor (4) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density, and stress.

Next, consider the electromagnetic fields coupled to the electric current J^{μ} of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate $P^{\mu}_{\rm EM}$ and $P^{\mu}_{\rm mat}$. Consequently, we should have

$$\partial_{\mu}T_{\text{net}}^{\mu\nu} = 0 \quad \text{for} \quad T_{\text{net}}^{\mu\nu} = T_{\text{EM}}^{\mu\nu} + T_{\text{mat}}^{\mu\nu} \tag{6}$$

but generally $\partial_{\mu}T^{\mu\nu}_{\rm EM} \neq 0$ and $\partial_{\mu}T^{\mu\nu}_{\rm mat} \neq 0$.

(d) Use Maxwell's equations to show that

$$\partial_{\mu}T^{\mu\nu}_{\rm EM} = -F^{\nu\lambda}J_{\lambda} \tag{7}$$

(in c = 1 units), and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_{λ} according to

$$\partial_{\mu}T_{\rm mat}^{\mu\nu} = +F^{\nu\lambda}J_{\lambda}.$$
 (8)

(e) Rewrite eq. (7) in non-relativistic notations and explain its physical meaning in terms of the electromagnetic energy, momentum, work, and forces.

2. Continuing problem 1, consider the EM fields coupled to a specific model of charged matter, namely a complex scalar field $\Phi(x) \neq \Phi^*(x)$ of electric charge $q \neq 0$. Altogether, the net Lagrangian for the A^{μ} , Φ , and Φ^* fields is

$$\mathcal{L}_{\rm net} = D^{\mu} \Phi^* D_{\mu} \Phi - m^2 \Phi^* \Phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$
(9)

where

$$D_{\mu}\Phi = (\partial_{\mu} + iqA_{\mu})\Phi$$
 and $D_{\mu}\Phi^* = (\partial_{\mu} - iqA_{\mu})\Phi^*$ (10)

are the *covariant* derivatives.

(a) Write down the equation of motion for all fields in a covariant from. Also, write down the electric current

$$J^{\mu} \stackrel{\text{def}}{=} -\frac{\partial \mathcal{L}}{\partial A_{\mu}} \tag{11}$$

in a manifestly gauge-invariant form and verify its conservation, $\partial_{\mu}J^{\mu} = 0$ (as long as the scalar fields satisfy their equations of motion).

(b) Write down the Noether stress-energy tensor for the whole system and show that

$$T_{\rm net}^{\mu\nu} \equiv T_{\rm EM}^{\mu\nu} + T_{\rm mat}^{\mu\nu} = T_{\rm Noether}^{\mu\nu} + \partial_{\lambda} \mathcal{K}^{\lambda\mu\nu}, \qquad (12)$$

where $T_{\rm EM}^{\mu\nu}$ is exactly as in eq. (4) for the free EM fields, the improvement tensor $\mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu}$ is also exactly as in eq. (5), and

$$T_{\rm mat}^{\mu\nu} = D^{\mu}\Phi^* D^{\nu}\Phi + D^{\nu}\Phi^* D^{\mu}\Phi - g^{\mu\nu} (D_{\lambda}\Phi^* D^{\lambda}\Phi - m^2\Phi^*\Phi).$$
(13)

Note: although the improvement tensor $\mathcal{K}^{\lambda\mu\nu}$ for the EM + matter system is the same as for the free EM fields, in presence of an electric current J^{μ} its derivative $\partial_{\lambda}\mathcal{K}^{\lambda\mu\nu}$ contains an extra $J^{\mu}A^{\nu}$ term. Pay attention to this term — it is important for obtaining the gauge-invariant stress-energy tensor (13) for the scalar field.

(c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_{\mu}, D_{\nu}]\Phi = iqF_{\mu\nu}\Phi, \qquad [D_{\mu}, D_{\nu}]\Phi^* = -iqF_{\mu\nu}\Phi^*$$
(14)

to show that

$$\partial_{\mu}T_{\rm mat}^{\mu\nu} = +F^{\nu\lambda}J_{\lambda} \tag{15}$$

exactly as in eq. (8), and therefore the *net* stress-energy tensor (12) is conserved, *cf.* problem $\mathbf{1}(d)$.

 In class, I have focused on the *fundamental multiplet* of the local SU(N) symmetry, *i.e.*, a set of N fields (complex scalars or Dirac fermions) which transform as a complex N-vector,

$$\Psi'(x) = U(x)\Psi(x) \quad i.e. \quad \Psi'_i(x) = \sum_j U_i^{\ j}(x)\Psi_j(x), \quad i,j = 1, 2, \dots, N$$
(16)

where U(x) is an *x*-dependent unitary $N \times N$ matrix, det $U(x) \equiv 1$. Now consider $N^2 - 1$ real fields $\Phi^a(x)$ forming an *adjoint multiplet*: In matrix form

$$\Phi(x) = \sum_{a} \Phi^{a}(x) \times \frac{\lambda^{a}}{2}$$
(17)

is a traceless hermitian $N \times N$ matrix which transforms under the local SU(N) symmetry as

$$\Phi'(x) = U(x)\Phi(x)U^{\dagger}(x).$$
(18)

Note that this transformation law preserves the $\Phi^{\dagger} = \Phi$ and the tr(Φ) = 0 conditions. The covariant derivatives D_{μ} act on an adjoint multiplet according to

$$D_{\mu}\Phi(x) = \partial_{\mu}\Phi(x) + i[\mathcal{A}_{\mu}(x), \Phi(x)] \equiv \partial_{\mu}\Phi(x) + i\mathcal{A}_{\mu}(x)\Phi(x) - i\Phi(x)\mathcal{A}_{\mu}(x),$$
(19)

or in components

$$D_{\mu}\Phi^{a}(x) = \partial_{\mu}\Phi_{a}(x) - f^{abc}\mathcal{A}^{b}_{\mu}(x)\Phi^{c}(x).$$
(20)

(a) Verify that these derivatives are indeed covariant under finite gauge transforms.

(b) Verify the Leibniz rule for covariant derivatives of matrix products. Let $\Phi(x)$ and $\Xi(x)$ be two adjoint multiplets while $\Psi(x)$ is a fundamental multiplet and $\Psi^{\dagger}(x)$ is its hermitian conjugate (row vector of Ψ_i^*). Show that

$$D_{\mu}(\Phi\Xi) = (D_{\mu}\Phi)\Xi + \Phi(D_{\mu}\Xi),$$

$$D_{\mu}(\Phi\Psi) = (D_{\mu}\Phi)\Psi + \Phi(D_{\mu}\Psi),$$

$$D_{\mu}(\Psi^{\dagger}\Xi) = (D_{\mu}\Psi^{\dagger})\Xi + \Psi^{\dagger}(D_{\mu}\Xi).$$
(21)

(c) Show that for an adjoint multiplet $\Phi(x)$,

$$[D_{\mu}, D_{\nu}]\Phi(x) = i[\mathcal{F}_{\mu\nu}(x), \Phi(x)] = ig[F_{\mu\nu}(x), \Phi(x)]$$
(22)

or in components $[D_{\mu}, D_{\nu}]\Phi^{a}(x) = -gf^{abc}F^{b}_{\mu\nu}(x)\Phi^{c}(x).$

• In my notations A_{μ} and $F_{\mu\nu}$ are canonically normalized fields while $\mathcal{A}_{\mu} = gA_{\mu}$ and $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$ are normalized by the symmetry action.

In class, I have argued (using covariant derivatives) that the tension fields $\mathcal{F}_{\mu\nu}(x)$ themselves transform according to eq. (18). In other words, the $\mathcal{F}^{a}_{\mu\nu}(x)$ form an adjoint multiplet of the SU(N) symmetry group.

- (d) Verify the $\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^{\dagger}(x)$ transformation law directly from the definition $\mathcal{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_{\mu}\mathcal{A}_{\nu} \partial_{\nu}\mathcal{A}_{\mu} + i[\mathcal{A}_{\mu}, \mathcal{A}_{\nu}]$ and the non-abelian gauge transform of the \mathcal{A}_{μ} fields.
- (e) Verify the covariant differential identity for the non-abelian tension fields $\mathcal{F}_{\mu\nu}(x)$:

$$D_{\lambda}\mathcal{F}_{\mu\nu} + D_{\mu}\mathcal{F}_{\nu\lambda} + D_{\nu}\mathcal{F}_{\lambda\mu} = 0.$$
⁽²³⁾

Note the covariant derivatives (19) in this equation.

Finally, consider the SU(N) Yang–Mills theory — the non-abelian gauge theory that does not have any fields except $\mathcal{A}^{a}(x)$ and $\mathcal{F}^{a}(x)$; its Lagrangian is

$$\mathcal{L}_{\rm YM} = -\frac{1}{2g^2} \operatorname{tr} \left(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right) = \sum_a \frac{-1}{4} F^a_{\mu\nu} F^{a\mu\nu}.$$
(24)

(f) Show that the Euler–Lagrange field equations for the Yang–Mills theory can be written in covariant form as $D_{\mu}\mathcal{F}^{\mu\nu} = 0$.

Hint: first show that for an infinitesimal variation $\delta \mathcal{A}_{\mu}(x)$ of the non-abelian gauge fields, the tension fields vary according to $\delta \mathcal{F}_{\mu\nu}(x) = D_{\mu} \delta \mathcal{A}_{\nu}(x) - D_{\nu} \delta \mathcal{A}_{\mu}(x)$.

4. Continuing the previous problem, consider an SU(N) gauge theory in which $N^2 - 1$ vector fields $A^a_{\mu}(x)$ interact with some "matter" fields $\phi_{\alpha}(x)$,

$$\mathcal{L} = -\frac{1}{2g^2} \operatorname{tr} \left(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right) + \mathcal{L}_{\mathrm{mat}}(\phi, D_{\mu}\phi).$$
(25)

For the moment, let me keep the matter fields completely generic — they can be scalars, or vectors, or spinors, or whatever, and form any kind of a multiplet of the local SU(N) symmetry as long as such multiplet is complete and non-trivial. All we need to know right now is that there are well-defined covariant derivatives $D_{\mu}\phi$ that depend on the gauge fields A^a_{μ} , which give rise to the currents

$$J^{a\mu} = -\frac{\partial \mathcal{L}_{\text{mat}}}{\partial A^a_{\mu}} = -\sum_{\phi} \frac{\partial \mathcal{L}_{\text{mat}}}{\partial (D_{\mu}\phi)} \times ig\hat{T}^a\phi.$$
(26)

Collectively, these $N^2 - 1$ currents should form an adjoint multiplet $J^{\mu} = \sum_{a} (\frac{1}{2}\lambda^a) J^{a\mu}$ of the SU(N) symmetry.

(a) Show that in this theory the equation of motion for the A^a_μ fields are $D_\mu F^{a\mu\nu} = J^{a\nu}$ and that consistency of these equations requires require the currents to be *covariantly conserved*,

$$D_{\mu}J^{\mu} = \partial_{\mu}J^{\mu} + i[\mathcal{A}_{\mu}, J^{\mu}] = 0, \qquad (27)$$

or in components, $\partial_{\mu}J^{a\mu} - f^{abc}\mathcal{A}^{b}_{\mu}J^{c\mu} = 0.$

Note: a covariantly conserved current does not lead to a conserved charge,

$$(d/dt)\int d^3\mathbf{x} J^{a0}(\mathbf{x},t)\neq 0!$$

Now consider a simple example of matter fields — a fundamental multiplet $\Psi(x)$ of N scalar fields $\Psi^{i}(x)$, with a Lagrangian

$$\mathcal{L}_{\text{mat}} = D_{\mu}\Psi^{\dagger}D^{\mu}\Psi - m^{2}\Psi^{\dagger}\Psi - \frac{\lambda}{4}(\Psi^{\dagger}\Psi)^{2}, \qquad \mathcal{L}_{\text{net}} = \mathcal{L}_{\text{mat}} - \frac{1}{2g^{2}}\operatorname{tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}).$$
(28)

(b) Derive the SU(N) currents $J^{a\mu}$ for this set of fields and verify that under SU(N)

symmetries the currents transform covariantly into each other as members of the adjoint multiplet. That is, the $N \times N$ matrix $J^{\mu} = \sum_{a} (\frac{1}{2}\lambda^{a}) J^{a\mu}$ transforms according to eq. (18).

Hint: for any complex vectors Ψ and Ψ' , $\sum_{a} (\Psi^{\dagger} \lambda^{a} \Psi') \lambda^{a} = 2\Psi' \otimes \Psi^{\dagger} - \frac{2}{N} (\Psi^{\dagger} \Psi') \times \mathbf{1}$.

(c) Finally, verify the covariant conservation $D_{\mu}J^{a\mu}$ of these currents when the scalar fields $\Psi^{i}(x)$ and $\Psi^{\dagger}_{i}(x)$ obey their equations of motion.