1. Consider the continuous Lorentz group $SO^+(3,1)$ and its generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$. In 3D terms, the six independent $\hat{J}^{\mu\nu}$ generators comprise the 3 components of the angular momentum $\hat{J}^i = \frac{1}{2}\epsilon^{ijk}\hat{J}^{jk}$ — which generate the rotations of space — plus 3 generators $\hat{K}^i = \hat{J}^{0i} = -\hat{J}^{i0}$ of the Lorentz boosts.

   (a) In 4D terms, the commutation relations of the Lorentz generators are

$$\left[\hat{J}^{\alpha\beta}, \hat{J}^{\mu\nu}\right] = ig^{\beta\mu}\hat{J}^{\alpha\nu} - ig^{\alpha\mu}\hat{J}^{\beta\nu} - ig^{\beta\nu}\hat{J}^{\alpha\mu} + ig^{\alpha\nu}\hat{J}^{\beta\mu}. \tag{1}$$

   Show that in 3D terms, these relations become

$$\left[\hat{J}^i, \hat{J}^j\right] = i\epsilon^{ijk}\hat{J}^k, \quad \left[\hat{J}^i, \hat{K}^j\right] = i\epsilon^{ijk}\hat{K}^k, \quad \left[\hat{K}^i, \hat{K}^j\right] = -i\epsilon^{ijk}\hat{J}^k. \tag{2}$$

   The Lorentz symmetry dictates the commutation relations of the $\hat{J}^{\mu\nu}$ with any operators comprising a Lorentz multiplet. In particular, for any Lorentz vector $\hat{V}^\mu$

$$\left[\hat{V}^\lambda, \hat{J}^{\mu\nu}\right] = ig^{\lambda\mu}\hat{V}^\nu - ig^{\lambda\nu}\hat{V}^\mu. \tag{3}$$

   (b) Spell out these commutation relations in 3D terms, then use them to show that the Lorentz boost generators $\hat{\mathbf{K}}$ do not commute with the Hamiltonian $\hat{H}$.

   (c) Show that even in the non-relativistic limit, the Galilean boosts $t' = t$, $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ and their generators $\hat{\mathbf{K}}_G$ do not commute with the Hamiltonian.

   Note: Only the *time-independent* symmetries commute with the Hamiltonian. But when the action of a symmetry is manifestly time dependent — like a Galilean boost $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ or a Lorentz boost — the symmetry operators do not commute with the time evolution and hence with the Hamiltonian.

2. Next, consider the little group $G(p)$ of Lorentz symmetries preserving some momentum 4–vector p^μ. For the moment, allow the p^μ to be time-like, light-like, or even space-like — anything goes as long as $p \neq 0$.

   (a) Show that the little group $G(p)$ is generated by the 3 components of the vector

$$\hat{\mathbf{R}} \;=\; p^0\hat{\mathbf{J}} \;+\; \mathbf{p}\times\hat{\mathbf{K}} \tag{4}$$

   after a suitable component-by-component rescaling.

   Suppose the momentum p^μ belongs to a massive particle, thus $p^\mu p_\mu = m^2 > 0$. For simplicity, assume the particle moves in z direction with velocity β, thus $p^\mu = (E,0,0,p)$ for $E=\gamma m$ and $p=\beta\gamma m$. In this case, the properly normalized generators of the little group $G(p)$ are the

$$\begin{aligned}
\widetilde{J}^x \;&=\; \frac{1}{m}\,\hat{R}^x \;=\; \gamma\hat{J}^x \;-\; \beta\gamma\hat{K}^y,\\
\widetilde{J}^y \;&=\; \frac{1}{m}\,\hat{R}^y \;=\; \gamma\hat{J}^y \;+\; \beta\gamma\hat{K}^x,\\
\widetilde{J}^z \;&=\; \frac{1}{\gamma m}\,\hat{R}^z \;=\; \hat{J}^z, \quad \text{the helicity.}
\end{aligned} \tag{5}$$

   (b) Show that these generators have angular-momentum-like commutators with each other, $[\widetilde{J}^i,\widetilde{J}^j] = i\epsilon^{ijk}\widetilde{J}^k$. Consequently, the little group $G(p)$ is isomorphic to the rotation group $SO(3)$.

   Now suppose the momentum p^μ belongs to a massless particle, $p^\mu p_\mu = 0$. Again, assume for simplicity that the particle moves in the z direction, thus $p^\mu = (E,0,0,E)$. In this case, we cannot normalize the generators of the little group as in eq. (5); instead, let's normalize them according to

$$\hat{\mathbf{I}} \;=\; \frac{1}{E}\hat{\mathbf{R}} \;=\; \hat{\mathbf{J}} \;+\; \vec{\beta}\times\hat{\mathbf{K}}, \tag{6}$$

   or in components,

$$\hat{I}^x \;=\; \hat{J}^x - \hat{K}^y, \quad \hat{I}^y \;=\; \hat{J}^y + \hat{K}^x, \quad \hat{I}^z \;=\; \hat{J}^z. \tag{7}$$

   (c) Show that these generators obey similar commutation relations to the $\hat{p}^x$, $\hat{p}^y$, and $\hat{J}^z$

operators, namely

$$[\hat{J}^z, \hat{I}^x] \;=\; +i\hat{I}^y, \quad [\hat{J}^z, \hat{I}^y] \;=\; -i\hat{I}^x, \quad [\hat{I}^x, \hat{I}^y] \;=\; 0. \tag{8}$$

Consequently, the little group $G(p)$ is isomorphic to the ISO(2) group of *rotations and translations* in the xy plane.

(d) Finally, show that for a tachyonic momentum with $p^\mu p_\mu < 0$, the properly normalized generators of the little group have similar commutation relations to the $\hat{K}^x$, $\hat{K}^y$, and $\hat{J}^z$ operators. Consequently, the little group $G(p)$ is isomorphic to the $SO^+(2,1)$, the continuous Lorentz group in $2+1$ spacetime dimensions.

3. Now let's focus on the massless particles. As explained in class, the finite unitary multiplets of the $G(p) \cong \mathrm{ISO}(2)$ group generated by the (7) operators are singlets $|\lambda\rangle$, although they are non-trivial singlets for $\lambda \neq 0$. Specifically, the state $|\lambda\rangle$ is an eigenstate of the helicity operator $\hat{J}^z$ (for the momentum in the z direction) and are annihilated by the $\hat{I}^{x,y}$ operators,

$$\hat{J}^z\,|\lambda\rangle \;=\; \lambda\,|\lambda\rangle\,, \quad \hat{I}^x\,|\lambda\rangle \;=\; 0, \quad \hat{I}^y\,|\lambda\rangle \;=\; 0. \tag{9}$$

(a) Show that in 4D terms, the state $|p,\lambda\rangle$ of a *massless* particle satisfies

$$\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\,|p,\lambda\rangle \;=\; \lambda\hat{P}_\mu\,|p,\lambda\rangle\,. \tag{10}$$

(b) Use eq. (10) to show that *continuous* Lorentz transforms do not change helicities of *massless* particles,

$$\forall L \in \mathrm{SO}^+(3,1), \quad \widehat{\mathcal{D}}(L)\,|p,\lambda\rangle \;=\; |Lp, \text{same}\,\lambda\rangle \times e^{i\,\text{phase}}. \tag{11}$$

4. Finally, consider various relations between the $SO^+(3,1)$ Lorentz group and its double cover $\mathrm{Spin}(3,1) = SL(2,\mathbf{C})$.

   (a) Show that the components of the two 3–vectors

$$\hat{\mathbf{J}}_+ \;=\; \tfrac{1}{2}\big(\hat{\mathbf{J}} + i\hat{\mathbf{K}}\big) \quad\text{and}\quad \hat{\mathbf{J}}_- \;=\; \tfrac{1}{2}\big(\hat{\mathbf{J}} - i\hat{\mathbf{K}}\big) \;=\; \hat{\mathbf{J}}_+^\dagger. \tag{12}$$

   obey commutation relations

$$\big[\hat{J}_+^i, \hat{J}_+^j\big] \;=\; i\epsilon^{ijk}\hat{J}_+^k, \quad \big[\hat{J}_-^i, \hat{J}_-^j\big] \;=\; i\epsilon^{ijk}\hat{J}_-^k, \quad\text{but}\quad \big[\hat{J}_+^i, \hat{J}_-^j\big] \;=\; 0. \tag{13}$$

   (b) Let $M = M_{\mathbf{2}}(L)$ and $\overline{M} = M_{\overline{\mathbf{2}}}(L)$ be matrices representing the same continuous Lorentz symmetry $L \in SO^+(3,1)$ in the $\mathbf{2}$ and the $\overline{\mathbf{2}}$ spinor representations. Use eqs. (33) and (34) of my notes on Lorentz representations to show that

$$\overline{M} = \sigma_2 M^* \sigma_2 \quad\text{and}\quad M \;=\; \sigma_2 \overline{M}^* \sigma_2\,. \tag{14}$$

   Hint: prove and use $\sigma_2 \boldsymbol{\sigma}^* \sigma_2 = -\boldsymbol{\sigma}$.

Now consider the vector representation of the Lorentz symmetry and the equivalent bi-spinor representation of the $SL(2,\mathbf{C})$. In the matrix form, the ($j_+ = j_- = \frac{1}{2}$) bi-spinor multiplet of $SL(2,\mathbf{C})$ is a complex 2×2 matrix V which transforms according to

$$V' \;=\; M \times V \times M^\dagger \quad \text{for } M \in SL(2,\mathbf{C}). \tag{15}$$

Let's identify this bi-spinor with a Lorentz vector V^μ according to

$$V \;=\; V^\mu \sigma_\mu \;=\; V^0\, \mathbf{1}_{2\times 2} \;+\; \mathbf{V}\cdot\boldsymbol{\sigma}, \tag{16}$$

where $\sigma_\mu = (1, \boldsymbol{\sigma})$. (Note the downstairs index of σ_μ; for an upstairs index we have $\sigma^\mu = (1, -\boldsymbol{\sigma})$.) The bi-spinor transform (15) defines a linear transform $V'^\mu = L^\mu_{\ \nu} V^\nu$ of the vector V^μ.

(c) Show that this transform is real (real V'^μ for real V^ν) and Lorentzian (preserves $V'^\mu V_\mu = V^\nu V_\nu$).

Hint: show that $\det(V) = V_\mu V^\mu$.

(d) Show that the Lorentz transform $V'^\mu = L^\mu_{\ \nu} V^\nu$ is orthochronous.

For extra challenge, show that it is continuous, $L \in SO^+(3,1)$.

(e) Verify that this $SL(2,\mathbf{C}) \to SO^+(3,1)$ map respects the group law, $L^\mu_{\ \nu}(M_2 M_1) = L^\mu_{\ \lambda}(M_2) L^\lambda_{\ \nu}(M_1)$.

Finally, consider the tensor representations of the Lorentz symmetry.

(f) Show that the $(j_+ = 1, j_- = 1)$ representation is equivalent to a 2–index symmetric traceless tensor, $T^{\mu\nu} = T^{\nu\mu}$, $g_{\mu\nu} T^{\mu\nu} = 0$.

Also, show that the reducible $(j_+ = 1, j_- = 0) + (j_+ = 0, j_- = 1)$ representation is equivalent to a 2-index antisymmetric tensor, $F^{\mu\nu} = -F^{\nu\mu}$.

Hint: For any kind of angular momentum — Hermitian or not, — the tensor product of two doublets is a triplet plus a singlet, $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0)$.