- 1. Consider the continuous Lorentz group $SO^+(3,1)$ and its generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$. In 3D terms, the six independent $\hat{J}^{\mu\nu}$ generators comprise the 3 components of the angular momentum $\hat{J}^i = \frac{1}{2} \epsilon^{ijk} \hat{J}^{jk}$ — which generate the rotations of space — plus 3 generators $\hat{K}^i = \hat{J}^{0i} = -\hat{J}^{i0}$ of the Lorentz boosts.
 - (a) In 4D terms, the commutation relations of the Lorentz generators are

$$\left[\hat{J}^{\alpha\beta},\hat{J}^{\mu\nu}\right] = ig^{\beta\mu}\hat{J}^{\alpha\nu} - ig^{\alpha\mu}\hat{J}^{\beta\nu} - ig^{\beta\nu}\hat{J}^{\alpha\mu} + ig^{\alpha\nu}\hat{J}^{\beta\mu}.$$
 (1)

Show that in 3D terms, these relations become

$$\left[\hat{J}^{i},\hat{J}^{j}\right] = i\epsilon^{ijk}\hat{J}^{k}, \quad \left[\hat{J}^{i},\hat{K}^{j}\right] = i\epsilon^{ijk}\hat{K}^{k}, \quad \left[\hat{K}^{i},\hat{K}^{j}\right] = -i\epsilon^{ijk}\hat{J}^{k}.$$
 (2)

The Lorentz symmetry dictates the commutation relations of the $\hat{J}^{\mu\nu}$ with any operators comprising a Lorentz multiplet. In particular, for any Lorentz vector \hat{V}^{μ}

$$\left[\hat{V}^{\lambda}, \hat{J}^{\mu\nu}\right] = ig^{\lambda\mu}\hat{V}^{\nu} - ig^{\lambda\nu}\hat{V}^{\mu}.$$
(3)

- (b) Spell out these commutation relations in 3D terms, then use them to show that the Lorentz boost generators $\hat{\mathbf{K}}$ do not commute with the Hamiltonian \hat{H} .
- (c) Show that even in the non-relativistic limit, the Galilean boosts t' = t, $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ and their generators $\hat{\mathbf{K}}_G$ do not commute with the Hamiltonian.

Note: Only the *time-independent* symmetries commute with the Hamiltonian. But when the action of a symmetry is manifestly time dependent — like a Galilean boost $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ or a Lorentz boost — the symmetry operators do not commute with the time evolution and hence with the Hamiltonian.

- 2. Next, consider the little group G(p) of Lorentz symmetries preserving some momentum 4-vector p^{μ} . For the moment, allow the p^{μ} to be time-like, light-like, or even space-like anything goes as long as $p \neq 0$.
 - (a) Show that the little group G(p) is generated by the 3 components of the vector

$$\hat{\mathbf{R}} = p^0 \hat{\mathbf{J}} + \mathbf{p} \times \hat{\mathbf{K}}$$
(4)

after a suitable component-by-component rescaling.

Suppose the momentum p^{μ} belongs to a massive particle, thus $p^{\mu}p_{\mu} = m^2 > 0$. For simplicity, assume the particle moves in z direction with velocity β , thus $p^{\mu} = (E, 0, 0, p)$ for $E = \gamma m$ and $p = \beta \gamma m$. In this case, the properly normalized generators of the little group G(p) are the

$$\widetilde{J}^{x} = \frac{1}{m} \widehat{R}^{x} = \gamma \widehat{J}^{x} - \beta \gamma \widehat{K}^{y},
\widetilde{J}^{y} = \frac{1}{m} \widehat{R}^{y} = \gamma \widehat{J}^{y} + \beta \gamma \widehat{K}^{x},
\widetilde{J}^{z} = \frac{1}{\gamma m} \widehat{R}^{z} = \widehat{J}^{z}, \text{ the helicity.}$$
(5)

(b) Show that these generators have angular-momentum-like commutators with each other, $[\tilde{J}^i, \tilde{J}^j] = i\epsilon^{ijk}\tilde{J}^k$. Consequently, the little group G(p) is isomorphic to the rotation group SO(3).

Now suppose the momentum p^{μ} belongs to a massless particle, $p^{\mu}p_{\mu} = 0$. Again, assume for simplicity that the particle moves in the z direction, thus $p^{\mu} = (E, 0, 0, E)$. In this case, we cannot normalize the generators of the little group as in eq. (5); instead, let's normalize them according to

$$\hat{\mathbf{I}} = \frac{1}{E}\hat{\mathbf{R}} = \hat{\mathbf{J}} + \vec{\beta} \times \hat{\mathbf{K}}, \qquad (6)$$

or in components,

$$\hat{I}^x = \hat{J}^x - \hat{K}^y, \quad \hat{I}^y = \hat{J}^y + \hat{K}^x, \quad \hat{I}^z = \hat{J}^z.$$
 (7)

(c) Show that these generators obey similar commutation relations to the \hat{p}^x , \hat{p}^y , and \hat{J}^z

operators, namely

$$[\hat{J}^{z}, \hat{I}^{x}] = +i\hat{I}^{y}, \quad [\hat{J}^{z}, \hat{I}^{y}] = -i\hat{I}^{x}, \quad [\hat{I}^{x}, \hat{I}^{y}] = 0.$$
(8)

Consequently, the little group G(p) is isomorphic to the ISO(2) group of rotations and translations in the xy plane.

- (d) Finally, show that for a tachyonic momentum with $p^{\mu}p_{\mu} < 0$, the properly normalized generators of the little group have similar commutation relations to the \hat{K}^x , \hat{K}^y , and \hat{J}^z operators. Consequently, the little group G(p) is isomorphic to the $SO^+(2, 1)$, the continuous Lorentz group in 2 + 1 spacetime dimensions.
- 3. Now let's focus on the massless particles. As explained in class, the finite unitary multiplets of the $G(p) \cong \text{ISO}(2)$ group generated by the (7) operators are singlets $|\lambda\rangle$, although they are non-trivial singlets for $\lambda \neq 0$. Specifically, the state $|\lambda\rangle$ is an eigenstate of the helicity operator \hat{J}^z (for the momentum in the z direction) and are annihilated by the $\hat{I}^{x,y}$ operators,

$$\hat{J}^{z}|\lambda\rangle = \lambda |\lambda\rangle, \quad \hat{I}^{x}|\lambda\rangle = 0, \quad \hat{I}^{y}|\lambda\rangle = 0.$$
 (9)

(a) Show that in 4D terms, the state $|p, \lambda\rangle$ of a massless particle satisfies

$$\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^{\gamma}|p,\lambda\rangle = \lambda\hat{P}_{\mu}|p,\lambda\rangle.$$
(10)

(b) Use eq. (10) to show that *continuous* Lorentz transforms do not change helicities of massless particles,

$$\forall L \in \mathrm{SO}^+(3,1), \quad \widehat{\mathcal{D}}(L) | p, \lambda \rangle = | Lp, \operatorname{same} \lambda \rangle \times e^{i \operatorname{phase}}.$$
 (11)

- 4. Finally, consider various relations between the $SO^+(3,1)$ Lorentz group and its double cover $Spin(3,1) = SL(2, \mathbb{C})$.
 - (a) Show that the components of the two 3-vectors

$$\hat{\mathbf{J}}_{+} = \frac{1}{2} (\hat{\mathbf{J}} + i\hat{\mathbf{K}}) \text{ and } \hat{\mathbf{J}}_{-} = \frac{1}{2} (\hat{\mathbf{J}} - i\hat{\mathbf{K}}) = \hat{\mathbf{J}}_{+}^{\dagger}.$$
 (12)

obey commutation relations

$$\left[\hat{J}_{+}^{i},\hat{J}_{+}^{j}\right] = i\epsilon^{ijk}\hat{J}_{+}^{k}, \quad \left[\hat{J}_{-}^{i},\hat{J}_{-}^{j}\right] = i\epsilon^{ijk}\hat{J}_{-}^{k}, \quad \text{but} \quad \left[\hat{J}_{+}^{i},\hat{J}_{-}^{j}\right] = 0.$$
(13)

(b) Let $M = M_2(L)$ and $\overline{M} = M_{\overline{2}}(L)$ be matrices representing the same continuous Lorentz symmetry $L \in SO^+(3, 1)$ in the 2 and the $\overline{2}$ spinor representations. Use eqs. (33) and (34) of my notes on Lorentz representations to show that

$$\overline{M} = \sigma_2 M^* \sigma_2$$
 and $M = \sigma_2 \overline{M}^* \sigma_2$. (14)

Hint: prove and use $\sigma_2 \sigma^* \sigma_2 = -\sigma$.

Now consider the vector representation of the Lorentz symmetry and the equivalent bispinor representation of the $SL(2, \mathbb{C})$. In the matrix form, the $(j_+ = j_- = \frac{1}{2})$ bi-spinor multiplet of $SL(2, \mathbb{C})$ is a complex 2×2 matrix V which transforms according to

$$V' = M \times V \times M^{\dagger} \quad \text{for } M \in SL(2, \mathbf{C}).$$
(15)

Let's identify this bi-spinor with a Lorentz vector V^{μ} according to

$$V = V^{\mu} \sigma_{\mu} = V^0 \mathbf{1}_{2 \times 2} + \mathbf{V} \cdot \boldsymbol{\sigma}, \tag{16}$$

where $\sigma_{\mu} = (1, \boldsymbol{\sigma})$. (Note the downstairs index of σ_{μ} ; for an upstairs index we have $\sigma^{\mu} = (1, -\boldsymbol{\sigma})$.) The bi-spinor transform (15) defines a linear transform $V^{\prime \mu} = L^{\mu}_{\ \nu} V^{\nu}$ of the vector V^{μ} .

- (c) Show that this transform is real (real V'^{μ} for real V^{ν}) and Lorentzian (preserves $V'^{\mu}V_{\mu} = V^{\nu}V_{\nu}$). Hint: show that $\det(V) = V_{\mu}V^{\mu}$.
- (d) Show that the Lorentz transform $V'^{\mu} = L^{\mu}_{\ \nu} V^{\nu}$ is orthochronous. For extra challenge, show that it is continuous, $L \in SO^+(3, 1)$.
- (e) Verify that this $SL(2, \mathbb{C}) \to SO^+(3, 1)$ map respects the group law, $L^{\mu}_{\nu}(M_2M_1) = L^{\mu}_{\lambda}(M_2)L^{\lambda}_{\nu}(M_1).$

Finally, consider the tensor representations of the Lorentz symmetry.

(f) Show that the $(j_+ = 1, j_- = 1)$ representation is equivalent to a 2-index symmetric traceless tensor, $T^{\mu\nu} = T^{\nu\mu}$, $g_{\mu\nu}T^{\mu\nu} = 0$. Also, show that the reducible $(j_+ = 1, j_- = 0) + (j_+ = 0, j_- = 1)$ representation is

equivalent to a 2-index antisymmetric tensor, $F^{\mu\nu} = -F^{\nu\mu}$.

Hint: For any kind of angular momentum — Hermitian or not, — the tensor product of two doublets is a triplet plus a singlet, $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0)$.