1. Let's start with an exercise in Dirac matrices γ^μ. In this problem, you should not assume any explicit matrices for the γ^μ but simply use the anticommutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{1}$$

When necessary, you may also assume that the Dirac matrices are 4×4, and the γ^0 matrix is hermitian while the $\gamma^1,\gamma^2,\gamma^3$ matrices are antihermitian, $(\gamma^0)^\dagger = +\gamma^0$ while $(\gamma^i)^\dagger = -\gamma^i$ for $i=1,2,3$.

(a) The original Dirac equation used $\beta=\gamma^0$ and $\alpha^i=\gamma^0\gamma^i$ (for $i=1,2,3$) instead of the γ^μ. Show that eqs. (1) are equivalent to requiring all 4 matrices β and α^i to anticommute with each other and to square to 1.

(b) Show that $\gamma^\alpha\gamma_\alpha=4$, $\gamma^\alpha\gamma^\nu\gamma_\alpha=-2\gamma^\nu$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha=4g^{\mu\nu}$,
and $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha=-2\gamma^\nu\gamma^\mu\gamma^\lambda$.
Hint: use $\gamma^\alpha\gamma^\nu=2g^{\nu\alpha}-\gamma^\nu\gamma^\alpha$ repeatedly.

(c) The electron field in the EM background obeys the *covariant Dirac equation* $\bigl(i\gamma^\mu D_\mu - m\bigr)\Psi(x) = 0$ where $D_\mu\Psi=\partial_\mu\Psi - ieA_\mu\Psi$. Show that this equation implies

$$\Bigl(D_\mu D^\mu + m^2 - eF_{\mu\nu}S^{\mu\nu}\Bigr)\Psi(x) = 0. \tag{2}$$

Besides the 4 Dirac matrices γ^μ, there is another useful matrix $\gamma^5 \stackrel{\rm def}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$.

(d) Show that the γ^5 anticommutes with each of the γ^μ matrices — $\gamma^5\gamma^\mu=-\gamma^\mu\gamma^5$ — and commutes with all the spin matrices, $\gamma^5 S^{\mu\nu}=+S^{\mu\nu}\gamma^5$.

(e) Show that the γ^5 is hermitian and that $(\gamma^5)^2=1$.

(f) Show that $\gamma^5=(i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and that $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]}=+24i\epsilon^{\kappa\lambda\mu\nu}\,\gamma^5$.

(g) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}=+6i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

(h) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]}=-2iS^{\mu\nu}$, $\gamma^5\gamma^\mu$, and γ^5.

* My conventions here are: $\epsilon^{0123} = -1$, $\epsilon_{0123} = +1$, $\gamma^{[\mu}\gamma^{\nu]} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$, $\gamma^{[\lambda}\gamma^{\mu}\gamma^{\nu]} = \gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda$, *etc.*

2. This is an optional exercise, for extra challenge. Let's generalize the Dirac matrices to spacetime dimensions $d \neq 4$. Such matrices always satisfy the Clifford algebra (1), but their sizes depend on d.

   Generalization of the γ^5 to d dimensions is $\Gamma = i^n \gamma^0\gamma^1 \cdots \gamma^{d-1}$, where the pre-factor $i^n = \pm i$ or ± 1 is chosen such that $\Gamma = \Gamma^\dagger$ and $\Gamma^2 = +1$.

   (a) *For even* d, Γ anticommutes with all the γ^μ. Prove this, then use this fact to show that there are 2^d independent products of the γ^μ matrices, and consequently the matrices should be $2^{d/2} \times 2^{d/2}$.

   (b) *For odd* d, Γ commutes with all the γ^μ — prove this. Consequently, one can set $\Gamma = +1$ or $\Gamma = -1$; the two choices lead to in-equivalent sets of the γ^μ.

   Classify the independent products of the γ^μ for odd d and show that their net number is 2^{d-1}; consequently, the matrices should be $2^{(d-1)/2} \times 2^{(d-1)/2}$.

3. Now let's go back to $d = 3 + 1$ and learn about the Weyl spinors and Weyl spinor fields. Since all the spin matrices $S^{\mu\nu}$ commute with the γ^5, for all *continuous* Lorentz symmetries $L^\mu_{\ \nu}$ their Dirac-spinor representations $M_D(L) = \exp\left(-\frac{i}{2}\Theta_{\alpha\beta}S^{\alpha\beta}\right)$ are block-diagonal in the eigenbasis of the γ^5. This makes the Dirac spinor Ψ a *reducible* multiplet of the continuous Lorentz group $SO^+(3,1)$ — it comprises two different irreducible 2-component spinor multiplets, called the left-handed Weyl spinor ψ_L and the right-handed Weyl spinor ψ_R.

   This decomposition becomes clear in the Weyl convention for the Dirac matrices where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \quad \text{where} \quad \begin{aligned} \sigma^\mu &\stackrel{\text{def}}{=} \left(\mathbf{1}_{2\times 2}, +\boldsymbol{\sigma}\right), \\ \overline{\sigma}^\mu &\stackrel{\text{def}}{=} \left(\mathbf{1}_{2\times 2}, -\boldsymbol{\sigma}\right). \end{aligned} \tag{3}$$

   I am sorry for the opposite convention for the σ^μ and $\overline{\sigma}^\mu$ from the previous homework, somehow I misread the Peskin & Schroeder convention.

In the Weyl convention (3), the γ^5 matrix is diagonal, specifically

$$\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}. \tag{4}$$

(a) Verify eq. (4).

(b) Write down explicitly matrices for the $S^{\mu\nu}$ matrices in the Weyl convention and show that

$$S^{\mu\nu} = \begin{pmatrix} S_L^{\mu\nu} & 0 \\ 0 & S_R^{\mu\nu} \end{pmatrix} \tag{5}$$

where $S_L^{\mu\nu} = S_{\mathbf{2}}^{\mu\nu}$ and $S_R^{\mu\nu} = S_{\overline{\mathbf{2}}}^{\mu\nu}$ are respectively the $\mathbf{2}$ and $\overline{\mathbf{2}}$ representations of the Lorentz generators.

In light of eqs. (5), the Dirac spinor is a reducible $\mathbf{2}+\overline{\mathbf{2}}$ multiplet of the Spin(3,1) Lorentz group, and for any continuous Lorentz transform L we have

$$M_D(L) = \begin{pmatrix} M_L(L) & 0 \\ 0 & M_R(L) \end{pmatrix} \quad \text{for } M_L(L) = M_{\mathbf{2}}(L) \text{ and } M_R(L) = M_{\overline{\mathbf{2}}}(L). \tag{6}$$

In particular, for a space rotation R through angle θ around axis $\mathbf{n}$,

$$M_L(R) = M_R(R) = \exp\bigl(-\tfrac{i}{2}\,\theta\mathbf{n}\cdot\boldsymbol{\sigma}\bigr), \tag{7}$$

while for a Lorentz boost B of rapidity r in the direction $\mathbf{n}$,

$$M_L(B) = \exp(-\tfrac{1}{2}\,r\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr) \quad \text{while} \quad M_R(B) = \exp(+\tfrac{1}{2}\,r\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr), \tag{8}$$

cf. eqs. (33) and (34) of my notes on Lorentz representations.

(c) The more familiar β and γ parameters of a Lorentz boost are related to the rapidity as

$$\beta \;=\; \tanh(r), \quad \gamma \;=\; \cosh(r), \quad \beta\gamma \;=\; \sinh(r). \tag{9}$$

Show that in terms of these parameters, eqs. (8) translate to

$$M_L(B) \;=\; \sqrt{\gamma} \times \sqrt{1 - \beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,, \qquad M_R(B) \;=\; \sqrt{\gamma} \times \sqrt{1 + \beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,. \tag{10}$$

In the Weyl convention for the Dirac matrices, the Dirac spinor field $\Psi(x)$ splits into the left-handed Weyl spinor field $\psi_L(x)$ and the right-handed Weyl spinor field $\psi_R(x)$ according to

$$\Psi_{\text{Dirac}}(x) \;=\; \begin{pmatrix} \psi_L(x), \\ \psi_R(x) \end{pmatrix} \quad \text{where} \quad \begin{aligned} \psi'_L(x') \;&=\; M_L(L)\psi_L(x), \\ \psi'_R(x') \;&=\; M_R(L)\psi_R(x). \end{aligned} \tag{11}$$

(d) Show that the hermitian conjugate of each Weyl spinor transforms equivalently to the other spinor. Specifically, the $\sigma_2 \times \psi^*_L(x)$ transforms under continuous Lorentz symmetries like the $\psi_R(x)$, while the $\sigma_2 \times \psi^*_R(x)$ transforms like the $\psi_L(x)$.

Note: the * superscript on a multi-component quantum field means hermitian conjugation of each component field but without transposing the components, thus

$$\psi_L \;=\; \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix}, \quad \psi^*_L \;=\; \begin{pmatrix} \psi^\dagger_{L1} \\ \psi^\dagger_{L2} \end{pmatrix}, \quad \text{while} \quad \psi^\dagger_L \;=\; (\,\psi^\dagger_{L1} \quad \psi^\dagger_{L2}\,), \tag{12}$$

and likewise for the ψ_R and its conjugates.

Hint: use problem 4(b) of the previous homework#6.

Next, consider the Dirac Lagrangian $\mathcal{L} = \overline{\Psi}(i\gamma^\mu\partial_\mu \,-\, m)\Psi$.

(e) Express this Lagrangian in terms of the Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$ (and their conjugates $\psi^\dagger_L(x)$ and $\psi^\dagger_R(x)$).

(f) Show that for $m = 0$ — and only for $m = 0$ — the two Weyl spinor fields become independent from each other.

4. Finally, consider the plane-wave solutions of the Dirac equation, $\Psi_\alpha(x) = u_\alpha \times e^{-ipx}$ and $\Psi_\alpha(x) = v_\alpha \times e^{+ipx}$ for some x-independent Dirac spinors $u_\alpha(p,s)$ and $v_\alpha(p,s)$.

   (a) Check that these waves indeed solve the Dirac equation provided $p^2 = m^2$ while

$$(\not{p} - m)u(p,s) \;=\; 0, \quad (\not{p} + m)v(p,s) \;=\; 0 \tag{13}$$

   where $\not{p}$ is the Dirac slash notation for the $\gamma^\mu p_\mu$. Likewise, for any Lorentz vector a^μ, we may write $\not{a}$ to denote $\gamma^\mu a_\mu$.

   By convention, we always take $E = p^0 = +\sqrt{\mathbf{p}^2 + m^2}$ — that's why we have both $e^{-ipx}u_\alpha$ and $e^{+ipx}v_\alpha$ types of wave — while the spinor coefficients $u(p,s)$ and $v(p,s)$ are normalized to

$$u^\dagger(p,s)u(p,s') \;=\; v^\dagger(p,s)v(p,s') \;=\; 2E\delta_{s,s'}\,. \tag{14}$$

   In this problem we shall write down explicit formulae for these spinors in the Weyl convention for the γ^μ matrices.

   (b) Show that for $\mathbf{p} = 0$,

$$u(\mathbf{p} = \mathbf{0}, s) \;=\; \begin{pmatrix} \sqrt{m}\,\xi_s \\ \sqrt{m}\,\xi_s \end{pmatrix} \tag{15}$$

   where ξ_s is a two-component $SO(3)$ spinor encoding the electron's spin state. The ξ_s are normalized to $\xi_s^\dagger \xi_{s'} = \delta_{s,s'}$.

   (c) For other momenta, $u(p,s) = M_D(\text{boost}) \times u(\mathbf{p} = 0, s)$ for the boost that turns $(m, \vec{0})$ into p^μ. Use eqs. (10) to show that

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\,\xi_s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\,\xi_s \end{pmatrix}. \tag{16}$$

   (d) Use similar arguments to show that

$$v(p,s) \;=\; \begin{pmatrix} +\sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \;=\; \begin{pmatrix} +\sqrt{p_\mu \sigma^\mu}\,\eta_s \\ -\sqrt{p_\mu \bar{\sigma}^\mu}\,\eta_s \end{pmatrix} \tag{17}$$

   where η_s are two-component $SO(3)$ spinors normalized to $\eta_s^\dagger \eta_{s'} = \delta_{s,s'}$.

Physically, the η_s should have opposite spins from ξ_s — the holes in the Dirac sea have opposite spins (as well as p^μ) from the missing negative-energy particles. Mathematically, this requires $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$; we may solve this condition by letting $\eta_s = \sigma_2 \xi_s^* = \pm i\xi_{-s}^*$.

(e) Check that $\eta_s = \sigma_2 \xi_s^* = \pm i\xi_{-s}^*$ indeed provides for the $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$, then show that this leads to

$$v(p,s) = \gamma^2 u^*(p,s) \quad \text{and} \quad u(p,s) = \gamma^2 v^*(p,s). \tag{18}$$

(f) Show that for the ultra-relativistic electrons or positrons of definite helicity $\lambda = \pm\frac{1}{2}$, the Dirac plane waves become *chiral* — *i.e.*, dominated by one of the two irreducible Weyl spinor components $\psi_L(x)$ or $\psi_R(x)$ of the Dirac spinor $\Psi(x)$, while the other component becomes negligible. Specifically,

$$\begin{aligned} u(p,-\tfrac{1}{2}) &\approx \sqrt{2E}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}, \qquad & u(p,+\tfrac{1}{2}) &\approx \sqrt{2E}\begin{pmatrix}0\\ \xi_R\end{pmatrix},\\ v(p,-\tfrac{1}{2}) &\approx -\sqrt{2E}\begin{pmatrix}0\\ \eta_L\end{pmatrix}, \qquad & v(p,+\tfrac{1}{2}) &\approx \sqrt{2E}\begin{pmatrix}\eta_R\\ 0\end{pmatrix}. \end{aligned} \tag{19}$$

Note that for the electron waves the helicity agrees with the chirality — they are both left or both right, — but for the positron waves the chirality is opposite from the helicity.

In the last part of the previous problem, we saw that for $m = 0$ the LH and the RH Weyl spinor fields decouple from each other. Now this exercise show us which particle modes comprise each Weyl spinor: *The $\psi_L(x)$ and its hermitian conjugate $\psi_L^\dagger(x)$ contain the left-handed fermions and the right-handed antifermions, while the $\psi_R(x)$ and the $\psi_R^\dagger(x)$ contain the right-handed fermions and the left-handed antifermions.*