1. First, a reading assignment: my notes on annihilation and Compton scattering.

   (a) The first 12 pages of my notes are about electron-positron annihilation. I went over them in class, but please re-read them *carefully* and pay attention to the algebra. Make sure you understand and can follow all the calculations.

   (b) The remaining 5 pages (12–17) are about the Compton scattering. I wish I could cover them in class as well, but I ran out of time, hence this assignment. SO read the notes and pay attention to the lab-frame kinematics as it's quite different from the center-of-mass frame kinematics we have used in other examples.

2. Next, consider a QED-like theory comprised of EM field $A^\mu(x)$, electron field $\Psi(x)$, and a real scalar field $\varphi(x)$. The φ field is neutral but it has Yukawa coupling g to the electron field $\Psi(x)$ — which also couples to the EM field $A^\mu(x)$ according to the usual QED rules. Altogether,

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{D}-m_e)\Psi + \;\left[\tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi \;-\; \tfrac{1}{2}M_s^2\varphi^2\right] \;+\; g\varphi\times\overline{\Psi}\Psi. \tag{1}$$

   The scalar particles S are much heavier than electrons or positrons, $M_s \gg m_e$. However, relativistic electron and positron colliding with each other at CM energy $E_{\text{c.m.}} > M_s$ may annihilate into one photon and one scalar particle, $e^- + e^+ \to \gamma + S$.

   (a) Draw tree diagrams for the $e^- + e^+ \to \gamma + S$ process and write down the tree-level matrix element $\langle\gamma + S|\,\mathcal{M}\,|e^- + e^+\rangle$.

   (b) Verify the Ward identity for the photon. Note: the Ward identity does not have to work for individual diagrams, but it must work for the net tree amplitude.

   (c) Sum $|\mathcal{M}|^2$ over the photon's polarizations and average over the fermion's spins. Show that

$$\overline{|\mathcal{M}|^2} \;\equiv\; \frac{1}{4}\sum_{s_-,s_+}\sum_{\lambda}|\mathcal{M}|^2 \;=\; e^2g^2\left(\frac{A_{11}}{(t-m_e^2)^2} \;+\; \frac{A_{22}}{(u-m_e^2)^2} \;+\; \frac{2\Re A_{12}}{(t-m_e^2)(u-m_e^2)}\right) \tag{2}$$

where

$$\begin{aligned} A_{11} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q} + m_e)\gamma^\mu(\not{p}_- + m_e)\gamma_\mu(\not{q} + m_e)\Big), \\ A_{22} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)\gamma^\mu(\tilde{\not{q}} + m_e)(\not{p}_- + m_e)(\tilde{\not{q}} + m_e)\gamma_\mu\Big), \\ A_{12} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q} + m_e)\gamma^\mu(\not{p}_- + m_e)(\tilde{\not{q}} + m_e)\gamma_\mu\Big). \end{aligned} \tag{3}$$

Since $M_s \gg m_e$, the initial electron and positron must be ultra-relativistic. So let's simplify our calculation by neglecting the electron's mass both in the traces (3) and in the denominators in eq. (2).

(d) Evaluate the Dirac traces (3) in the $m_e \approx 0$ approximation and express them in terms of the Mandelstam variables s, t, u. Show that

$$\text{for } m_e \approx 0, \quad A_{11} \approx A_{22} \approx tu, \quad A_{12} \approx (t - M_s^2)(u - M_s^2). \tag{4}$$

Note: because of the scalar's mass, the kinematic relations between various momentum products such as $(k_\gamma p_\mp)$ and between the Mandelstam's s, t, and u are different from the $e^+e^- \to \gamma\gamma$ annihilation.

(e) Finally, assemble the net $|\mathcal{M}|^2$ (in the $m_e \approx 0$ approximation), work out the kinematics in the CM frame, and calculate the partial cross-section

$$\frac{d\sigma(e^-e^+ \to \gamma S)}{d\Omega_{\text{c.m.}}}.$$

3. Now let's change the subject from QED (or rather QED+scalar theory) to the spontaneous symmetry braking.

When an *exact* symmetry of a quantum field theory is spontaneously broken down, it gives rise to exactly massless Goldstone bosons. But when the spontaneously broken symmetry was only approximate to begin with, the would-be Goldstone bosons are no longer exactly massless but only relatively light. The best-known examples of such pseudo-Goldstone bosons are the pi-mesons $\pi^\pm$ and π^0, which are indeed much lighter then other hadrons. The Quantum ChromoDynamics theory (QCD) of strong interactions has an approximate chiral isospin symmetry $SU(2)_L \times SU(2)_R \cong \mathrm{Spin}(4)$. This symmetry would be exact if the two lightest quark flavors u and d were massless; in real life, the masses m_u and m_d are small but non quite zero, and the symmetry is only approximate. Somehow (and people are still arguing how), the chiral isospin symmetry is spontaneously broken down to the ordinary isospin symmetry $SU(2) \cong \mathrm{Spin}(3)$, and the 3 generators of the broken $\mathrm{Spin}(4)/\mathrm{Spin}(3)$ give rise to 3 (pseudo) Goldstone bosons $\pi^\pm$ and π^0.

As a toy model of *approximate* $SO(N+1)$ symmetry spontaneously broken down to $SO(N)$, consider the *linear sigma model* of $N+1$ scalar fields ϕ_i with the Lagrangian

$$\mathcal{L} \;=\; \sum_i \tfrac{1}{2}(\partial_\mu\phi_i)^2 \;-\; \frac{\lambda}{8}\left(\sum_i \phi_i^2 \,-\, f^2\right)^2 \;+\; \beta\lambda f^2 \times \phi_{N+1}\,. \tag{5}$$

For $\beta = 0$ this Lagrangian has exact $O(N+1)$ symmetry, which would be spontaneously broken down to $O(N)$ by non-zero vacuum expectation values of the scalar fields. For a non-zero β, the last term in the Lagrangian (5) *explicitly* breaks the $O(N+1)$ symmetry, but for $\beta \ll f$ we may treat the $O(N+1)$ as *approximate* symmetry.

(a) Assume $\beta > 0$ and $\beta \ll f$. Show that the scalar potential of the linear sigma model has a unique minimum at

$$\langle\phi_1\rangle \;=\; \cdots\; \langle\phi_N\rangle \;=\; 0, \quad \left\langle\phi_{N+1}\right\rangle \;=\; f \;+\; \beta \;+\; O(\beta^2/f). \tag{6}$$

(b) Re-express the Lagrangian (5) in terms of the shifted fields

$$\sigma(x) \;=\; \phi_{N+1}(x) \;-\; \left\langle\phi_{N+1}\right\rangle, \qquad \pi^i(x) \;=\; \phi_i(x) \quad \text{for } i = 1,\ldots,N. \tag{7}$$

and show that the π^i fields are massive but much lighter than the σ field. Specifically, $M_\pi^2 \approx \lambda f \times \beta$ while $M_\sigma^2 \approx \lambda f(f + 3\beta) \approx \lambda f^2 \gg M_\pi^2$.

In QCD terms, $N = 3$, the three $\pi^{1,2,3}$ fields (or rather the $\pi^0 = \pi^3$ and the $\pi^\pm = (\pi^1 \pm i\pi^2)/\sqrt{2}$) correspond to the three pi-mesons of rather small mass $m_\pi \approx 140$ MeV, and the σ corresponds to the very broad sigma resonance at about 500 MeV.

(c) Spell out the cubic and the quartic couplings of the σ and π^i fields to each other and show that

$$(\text{cubic coupling})^2 \;=\; (\text{quartic coupling}) \times \left(M_\sigma^2 - M_\pi^2\right). \tag{8}$$

For $\beta = 0$ and hence $M_\pi^2 = 0$, these couplings are precisely as in problem 4 of homework set#9 (eq. (3)). Therefore — as we saw in that homework — for low-energy pions with

$E \ll M_\sigma$, the scattering amplitudes $\mathcal{M}(\pi^j+\pi^k \to \pi^\ell+\pi^m)$ become small as $O(\lambda E_{\rm cm}^2/M_\sigma^2)$ or smaller.

For small $\beta \neq 0$ and hence small but non-zero pion mass, the coupling relation (8) is slightly different from what we had in homework#9, so the several tree diagrams contributing to the scattering of low-energy pions do not quite cancel each other.

(d) Recalculate the pion scattering amplitudes to allow for eq. (8) for $M_\pi^2 > 0$. Basically, go over solutions to homework#9, parts 4(c–d), and correct a few formulae.

In particular, show that to the leading order in β, for $s, t, u \ll M_\sigma$,

$$\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) \;\approx\; \frac{1}{f^2}\begin{pmatrix} (s - m_\pi^2)\times\delta^{jk}\delta^{\ell m} \,+\, (t - m_\pi^2)\times\delta^{j\ell}\delta^{km} \\ +\, (u - m_\pi^2)\times\delta^{jm}\delta^{k\ell} \end{pmatrix}, \quad (9)$$

which does not vanish when any of the pion's momenta becomes small. Instead, for *slow* pions with $|\mathbf{p}| \ll m_\pi$, this amplitude becomes

$$\mathcal{M}(\pi^j+\pi^k \to \pi^\ell+\pi^m) \;\approx\; \left(3\delta^{jk}\delta^{\ell m} - \delta^{j\ell}\delta^{km} - \delta^{jm}\delta^{k\ell}\right)\times\left(\frac{m_\pi^2}{f^2} \approx \frac{\lambda\beta}{f}\right) \neq 0. \quad (10)$$