

This homework set has 4 problems. The first 3 problems are about spontaneous symmetry breaking and the Higgs mechanism. The last problem is about the $e^- + e^+ \rightarrow \mu^- + \mu^+$ process in the Standard Model, where the virtual particle in the s channel can be a photon, a X^0 , or a Higgs.

1. Let's start with the $SU(2)$ gauge symmetry coupled to a doublet of Higgs fields $\Phi^{1,2}(x)$,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + D_\mu\Phi^\dagger D^\mu\Phi - \frac{\lambda}{2}\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2. \quad (1)$$

As discussed in class (*cf.* [my notes on the Higgs mechanism](#), pages 8–10), the Higgs doublet develops a vacuum expectation value (VEV)

$$\langle\Phi\rangle = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2)$$

which completely breaks the $SU(2)$ gauge symmetry and gives the vector bosons equal masses $M_V = \frac{1}{2}gv$.

The reason all 3 vector fields get the same mass is a non-obvious global $SU(2)$ symmetry of the scalar fields. Indeed, the scalar potential is invariant under the $SO(4)$ symmetry which mixes the real and the imaginary parts of the scalar fields, and the $SO(4)$ or rather Spin(4) group happens to be isomorphic to $SU(2) \times SU(2)$. The gauge symmetry acts as one of these $SU(2)$ factors while the other $SU(2)$ factor of the $SO(4)$ remains a global symmetry.

To make both $SU(2)_{\text{local}}$ and $SU(2)_{\text{global}}$ symmetries manifest, consider a 2×2 matrix

$$W = \begin{pmatrix} +\Phi_2^* & +\Phi^1 \\ -\Phi_1^* & +\Phi^2 \end{pmatrix} \quad (3)$$

- (a) Show that $\sigma_2 W^* \sigma_2 = W$, and that any matrix obeying this skewed reality condition has form (3) for some complex Φ^1 and Φ^2 .

(b) Show that under a gauge transform of the $\Phi(x)$ doublet, the $W(x)$ matrix-valued field transforms in the similar manner,

$$\Phi'(x) = U(x)\Phi(x) \implies W'(x) = U(x)W(x) \quad \forall U(x) \in SU(2). \quad (4)$$

(c) Show that in terms of the matrix-valued field $W(x)$ the Lagrangian (1) becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2}\text{tr}(D_\mu W^\dagger D^\mu W) - \frac{\lambda}{8}(\text{tr}(W^\dagger W) - v^2)^2, \quad (5)$$

and that it is invariant under $SU(2)_{\text{local}} \times SU(2)_{\text{global}}$ symmetry which acts as

$$\begin{aligned} W'(x) &= U_L(x)W(x)U_G^\dagger, \\ A'_\mu(x) &= U_L(x)A_\mu(x)U_L^{-1}(x) + \frac{i}{g}(\partial_\mu U_L(x))U_L^{-1}(x), \\ &\text{local } U_L(x) \in SU(2), \text{ global } U_G \in SU(2). \end{aligned} \quad (6)$$

The vacuum expectation value (2) — which translates to

$$\langle W \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (7)$$

— is not invariant under the $SU(2)_{\text{global}}$ symmetry factor of (6). Nevertheless, there are no Goldstone bosons because $\langle W \rangle$ is invariant under a modified global $SU(2)$ symmetry:

- Any $U_G \in SU(2)$ accompanied by $U_L(x) = U_G \forall x$.

(d) Argue that the generators of this modified $SU(2)'_{\text{global}}$ symmetry are

$$T^a[SU(2)'_{\text{global}}] = T^a[SU(2)_{\text{global}}] + T^a[SU(2)_{\text{local}}] \quad (8)$$

and check that the VEV $\langle W \rangle$ is indeed invariant under the $SU(2)'_{\text{global}}$.

Since the global $SU(2)$ symmetry is modified (by mixing with the gauge symmetry) rather than spontaneously broken, there are no Goldstone bosons for this symmetry. Moreover, the modification (8) make the symmetry act non-trivially on the massive vector fields.

(e) Show that the massive vector fields form a triplet of the $SU(2)'_{\text{global}}$, and that's why they all have the same mass.

2. Next, a more complicated problem on symmetry breaking. Consider an $N \times N$ matrix $\Phi(x)$ of complex scalar fields $\Phi_j^i(x)$, $i, j = 1, \dots, N$. In matrix notations, the Lagrangian is

$$\mathcal{L} = \text{tr} \left(\partial^\mu \Phi^\dagger \partial_\mu \Phi \right) - V(\Phi^\dagger \Phi) \quad (9)$$

where the potential is

$$V = \frac{\alpha}{2} \text{tr} \left(\Phi^\dagger \Phi \Phi^\dagger \Phi \right) + \frac{\beta}{2} \left(\text{tr} \left(\Phi^\dagger \Phi \right) \right)^2 + m^2 \text{tr} \left(\Phi^\dagger \Phi \right). \quad (10)$$

- (a) Show that this theory has global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ acting as

$$\Phi(x) \rightarrow e^{i\theta} U_L \Phi(x) U_R^\dagger, \quad U_L, U_R \in SU(N). \quad (11)$$

For $\alpha = 0$ the theory would have a much larger continuous symmetry group $SO(2N^2)$ which mixes up the real and the imaginary parts of all the Φ_j^i regardless of their complex structure or the $N \times N$ matrix structure of Φ . But for $\alpha \neq 0$ this huge symmetry group is reduced to $G = SU(N)_L \times SU(N)_R \times U(1)$. Proving that the $\text{tr} \left(\Phi^\dagger \Phi \Phi^\dagger \Phi \right)$ term has no other continuous symmetries besides G is a hard exercise in group theory, which fortunately is beyond the scope of this homework.

From now on, we take $\alpha, \beta > 0$ but $m^2 < 0$. In this regime, V is minimized for non-zero vacuum expectation values $\langle \Phi \rangle \neq 0$ of the scalar fields.

- (b) Let $(\kappa_1, \dots, \kappa_N)$ be eigenvalues of the hermitian matrix $\Phi^\dagger \Phi$. Express the potential (10) in terms of these eigenvalues and show that the minimum lies at

$$\kappa_1 = \kappa_2 = \dots = \kappa_N = C^2 = \frac{-m^2}{\alpha + N\beta} > 0. \quad (12)$$

In terms of the matrix Φ , eq. (12) means $\Phi = C \times$ a unitary matrix. All such minima are related by symmetries (11) to $\Phi = C \times$ the unit matrix, so without loss of generality we

may assume that the vacuum lies at

$$\langle \Phi \rangle = C \times \mathbf{1}_{N \times N} \quad i.e. \quad \langle \Phi^i_j \rangle = C \times \delta^i_j. \quad (13)$$

- (c) Show that the symmetries (11) preserving these VEVs are limited to the $U_L = U_R \in SU(N)$ and $\theta = 0$. In other words, the $SU(N)_L \times SU(N)_R \times U(1)$ symmetry of the theory is spontaneously broken down to the $SU(N)_V$ subgroup.

Now that we know the vacuum state of the theory and its symmetry, let's find the particle spectrum of the theory.

- (d) Expand the scalar potential in powers of the $\delta\Phi(x) = \Phi(x) - \langle \Phi \rangle$ and the $\delta\Phi^\dagger(x)$,

$$V(\delta\Phi^\dagger, \delta\Phi) = \text{const} + V_1 + V_2 + V_3 + V_4. \quad (14)$$

Show that $V_1 = 0$ while

$$V_2 = \frac{\alpha C^2}{2} \text{tr}((\delta\Phi^\dagger + \delta\Phi)^2) + \frac{\beta C^2}{2} \text{tr}^2(\delta\Phi^\dagger + \delta\Phi). \quad (15)$$

- (e) Altogether, the N^2 complex scalar fields give rise to $2N^2$ particle species. Find the masses of all those particles from eq. (15).

Hint: Split the complex matrix Φ into its hermitian and antihermitian parts, and also into trace and traceless parts,

$$\delta\Phi(x) = \frac{\chi_1(x) + i\chi_2(x)}{\sqrt{2N}} \times \mathbf{1}_{N \times N} + \frac{\varphi_1(x) + i\varphi_2(x)}{\sqrt{2}} \quad (16)$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are traceless hermitian matrices (or rather matrix-valued fields) while $\chi_1(x)$ and $\chi_2(x)$ are ordinary real fields.

- (f) Finally, organize the $2N^2$ particles into multiplets of the unbroken $SU(N)_V$ symmetry and make sure that all members of each multiple have the same mass.

Also, check the Nambu–Goldstone theorem for this model — verify that for each *spontaneously broken* generator of the symmetry (11) there is a massless particle with similar quantum numbers WRT the unbroken $SU(N)_V$ subgroup.

3. In the previous problem we had the continuous global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ spontaneously broken down to its $H = SU(N)_V$ subgroup. Now let's gauge the entire $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry and work out the Higgs mechanism. Thus, consider a theory of N^2 complex scalar fields $\Phi_j^i(x)$ organized into an $N \times N$ matrix Φ , and $2N^2 - 1$ real vector fields $B_\mu(x)$, $L_\mu^a(x)$, and $R_\mu^a(x)$, the latter organized into traceless hermitian matrices $L_\mu(x) = \sum_a L_\mu^a(x) \times \frac{1}{2}\lambda^a$ and $R_\mu(x) = \sum_a R_\mu^a(x) \times \frac{1}{2}\lambda^a$, where $a = 1, \dots, (N^2 - 1)$ and λ^a are the Gell-Mann matrices of $SU(N)$. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}\text{tr}(L_{\mu\nu}L^{\mu\nu}) - \frac{1}{2}\text{tr}(R_{\mu\nu}R^{\mu\nu}) + \text{tr}\left(D^\mu\Phi^\dagger D_\mu\Phi\right) - V(\Phi^\dagger\Phi), \quad (17)$$

where

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ L_{\mu\nu} &= \partial_\mu L_\nu - \partial_\nu L_\mu + ig[L_\mu, L_\nu], \\ R_{\mu\nu} &= \partial_\mu R_\nu - \partial_\nu R_\mu + ig[R_\mu, R_\nu], \\ D_\mu\Phi &= \partial_\mu\Phi + ig'B_\mu\Phi + igL_\mu\Phi - ig\Phi R_\mu, \\ D_\mu\Phi^\dagger &= (D_\mu\Phi)^\dagger = \partial_\mu\Phi^\dagger - ig'B_\mu\Phi^\dagger + igR_\mu\Phi^\dagger - ig\Phi^\dagger L_\mu. \end{aligned} \quad (18)$$

For simplicity, I assume equal gauge couplings $g_L = g_R = g$ for the two $SU(N)$ factors of the gauge group, but the abelian coupling g' is different.

Finally, the scalar potential V is precisely as in the previous problem,

$$V = \frac{\alpha}{2}\text{tr}(\Phi^\dagger\Phi\Phi^\dagger\Phi) + \frac{\beta}{2}\text{tr}^2(\Phi^\dagger\Phi) + m^2\text{tr}(\Phi^\dagger\Phi), \quad \alpha, \beta > 0, \quad m^2 < 0, \quad (19)$$

hence similar VEVs of the scalar fields: up to a gauge symmetry,

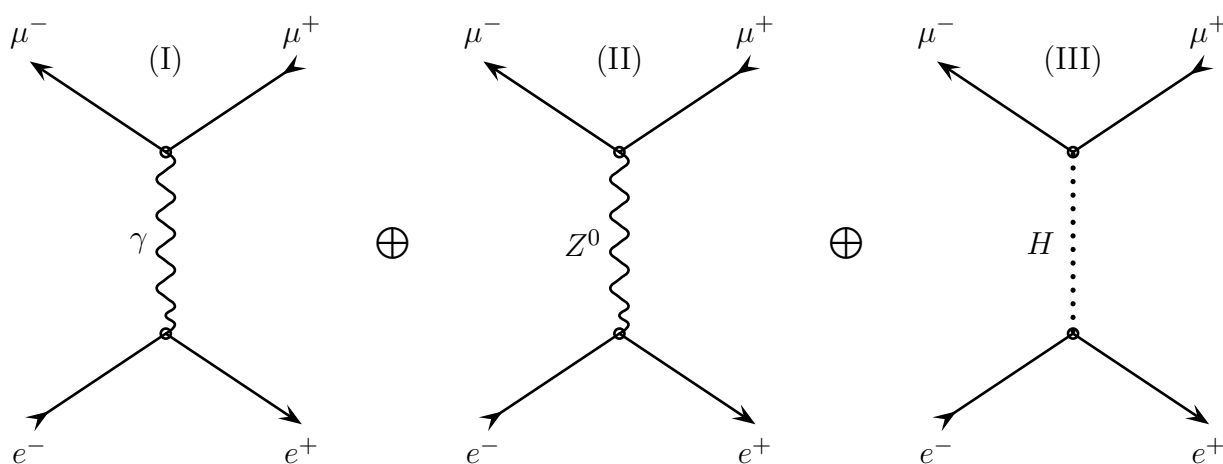
$$\langle\Phi\rangle = C \times \mathbf{1}_{N \times N} \quad \text{where} \quad C = \sqrt{\frac{-m^2}{\alpha + N\beta}}, \quad (20)$$

which breaks the $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry down to the $H = SU(N)_V$ subgroup.

- (a) The Higgs mechanism makes N^2 out of $2N^2 - 1$ vector fields massive. Calculate their masses by plugging $\langle \Phi \rangle$ for the $\Phi(x)$ into the $\text{tr}(D_\mu \Phi^\dagger D^\mu \Phi)$ term of the Lagrangian. In particular, show that the abelian gauge field B_μ and the $X_\mu^a = \frac{1}{\sqrt{2}}(L_\mu^a - R_\mu^a)$ combinations of the $SU(N)$ gauge fields become massive, while the $V_\mu^a = \frac{1}{\sqrt{2}}(L_\mu^a + R_\mu^a)$ combinations remain massless.
- (b) Find the effective Lagrangian for the massless vector fields $V_\mu^a(x)$ by freezing all the other fields, *i.e.* setting $B_\mu(x) \equiv 0$, $X_\mu^a(x) \equiv 0$, and $\Phi(x) \equiv \langle \Phi \rangle$. Show that this Lagrangian describes a Yang–Mills theory with gauge group $SU(N)_V$ and gauge coupling $g_V = g/\sqrt{2}$.
- ★ For extra challenge, allow for unequal gauge couplings $g_L \neq g_R$. Find which combinations of the $L_\mu^a(x)$ and $R_\mu^a(x)$ fields remain massless in this case, then derive the effective Lagrangian for these massless fields by freezing everything else. As in part (b), you should get an $SU(N)$ Yang–Mills theory, but this time the gauge coupling is

$$g_v = \frac{g_L g_R}{\sqrt{g_L^2 + g_R^2}}. \quad (21)$$

4. Finally, consider the $e^+e^- \rightarrow \mu^+\mu^-$ pair production in the Standard Model of electroweak interactions rather than in just QED. Unlike QED, the Standard Model has three diagrams contributing to this process: one with the virtual photon in the s channel, one with the virtual Z^0 gauge boson, and one with the virtual Higgs scalar,



center-of-mass frame.) I suggest you proceed exactly as in [homework set#10](#) (problem 2) for the \mathcal{M}_γ amplitude, but mind the γ^5 factors in the vertices and the massive vector propagator for the Z_0 . Your answer should have form

$$\mathcal{M}_Z = \pm F(s) \times \mathcal{M}_\gamma \quad (26)$$

where the \pm sign depends on the helicities while

$$F(s) = \left(\frac{e}{4g'} \right)^2 \frac{s}{s - M_Z^2 + i\Gamma_Z M_Z}. \quad (27)$$

- (d) Combine the amplitudes due to virtual Z and virtual photon and calculate the polarized partial cross-sections $d\sigma(e^-e^+ \rightarrow \mu^-\mu^+)/d\Omega$ as functions of EM energy² = s , scattering angle θ , and helicities of all 4 fermions involved. Specifically, show that

$$\begin{aligned} \frac{d\sigma(e_L^- + e_L^+ \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_R^+ \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = 0, \\ \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \rightarrow \mu_L^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \rightarrow \mu_R^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = 0. \end{aligned} \quad (28)$$

while

$$\begin{aligned} \frac{d\sigma(e_L^- + e_R^+ \rightarrow \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \rightarrow \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} \\ &= \frac{\alpha^2}{4s} \times |1 + F(s)|^2 \times (1 + \cos\theta)^2, \\ \frac{d\sigma(e_L^- + e_R^+ \rightarrow \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \rightarrow \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} \\ &= \frac{\alpha^2}{4s} \times |1 - F(s)|^2 \times (1 - \cos\theta)^2, \end{aligned} \quad (29)$$

where the $|1 \pm F(s)|^2$ factor stem from the interference between the virtual-photon and virtual- Z diagrams.

- (e) Finally, assume un-polarized electron and positron beams and a spin-blind muon detector. Calculate the total cross section $\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ and the forward-backward

asymmetry

$$A = \frac{\sigma(\theta < \pi/2) - \sigma(\theta > \pi/2)}{\sigma(\theta < \pi/2) + \sigma(\theta > \pi/2)} \quad (30)$$

as functions of the total energy $E_{\text{c.m.}}$.

Note: In QED, the tree-level pair production is symmetric with respect to $\theta \rightarrow \pi - \theta$; the asymmetry in the Standard Model arises from the interference between the virtual-photon and virtual- Z diagrams.