

1. Verify the integrals used by the Feynman's parameter trick and its generalizations:

$$\frac{1}{AB} = \int_0^1 \frac{d\xi}{[\xi A + (1 - \xi)B]^2}, \quad (\text{F.a})$$

$$\frac{1}{A^n B} = \int_0^1 \frac{n \xi^{n-1} d\xi}{[\xi A + (1 - \xi)B]^{n+1}}, \quad (\text{F.b})$$

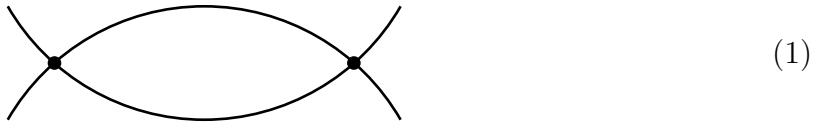
$$\frac{1}{A^n B^m} = \frac{(n + m - 1)!}{(n - 1)!(m - 1)!} \times \int_0^1 \frac{\xi^{n-1} (1 - \xi)^{m-1} d\xi}{[\xi A + (1 - \xi)B]^{n+m}}, \quad (\text{F.c})$$

$$\begin{aligned} \frac{1}{ABC} &= \int_0^1 d\xi \int_0^{1-\xi} \frac{2d\eta}{[\xi A + \eta B + (1 - \xi - \eta)C]^3} \\ &\equiv \iiint_{\xi, \eta, \zeta \geq 0} d\xi d\eta d\zeta \delta(\xi + \eta + \zeta - 1) \times \frac{2}{[\xi A + \eta B + \zeta C]^3}, \end{aligned} \quad (\text{F.d})$$

$$\frac{1}{A_1 A_2 \cdots A_k} = \int_{\xi_1, \dots, \xi_k \geq 0} \cdots \int d^k \xi \delta(\xi_1 + \cdots + \xi_k - 1) \times \frac{(k - 1)!}{[\xi_1 A_1 + \cdots + \xi_k A_k]^k}, \quad (\text{F.e})$$

$$\begin{aligned} \frac{1}{A_1^{n_1} A_2^{n_2} \cdots A_k^{n_k}} &= \frac{(n_1 + \cdots + n_k - 1)!}{(n_1 - 1)! \cdots (n_k - 1)!} \times \\ &\times \int_{\xi_1, \dots, \xi_k \geq 0} \cdots \int d^k \xi \delta(\xi_1 + \cdots + \xi_k - 1) \times \frac{\xi_1^{n_1-1} \cdots \xi_k^{n_k-1}}{[\xi_1 A_1 + \cdots + \xi_k A_k]^{(n_1 + \cdots + n_k)}}. \end{aligned} \quad (\text{F.f})$$

2. In class, we have evaluated the one-loop diagram



using the *hard-edge cutoff* as an ultraviolet regulator. Your task is to evaluate the same diagram using two other UV regulators: (1) Pauli-Villars, and (2) higher derivatives.

Show that all 3 regulators yield similar amplitudes of the form

$$\mathcal{M}(\text{diagram (1)}) = \frac{\lambda_{\text{bare}}^2}{32\pi^2} \times \left(\log \frac{\Lambda^2}{m^2} + C - J(t/m^2) + \text{negligible} \right) \quad (2)$$

where

$$J(t/m^2) = \int_0^1 d\xi \log \frac{m^2 - t \times \xi(1 - \xi)}{m^2}, \quad (3)$$

‘negligible’ stands for terms that vanish as negative powers of the cutoff scale Λ for $\Lambda \rightarrow \infty$, and C is an $O(1)$ numeric constant that depends on the particular UV regulator:

$$C_{\text{hard edge}} \neq C_{\text{Pauli Villars}} \neq C_{\text{higher derivative}}. \quad (4)$$

Fortunately, this regulator dependence can be canceled by adjusting the cutoff scale parameter Λ for each regulator: Let

$$\Lambda_{\text{HE}}^2 \times e^{C_{\text{HE}}} = \Lambda_{\text{PV}}^2 \times e^{C_{\text{PV}}} = \Lambda_{\text{HD}}^2 \times e^{C_{\text{HD}}}, \quad (5)$$

then all 3 regulators would yield exactly the same loop amplitude (2).

Note: the dimensional regularization also yields exactly the same amplitude (2), provided we identify the UV cutoff scale as

$$\Lambda_{\text{DR}}^2 = \mu^2 \times \exp\left(\frac{1}{\epsilon} = \frac{2}{4 - D}\right) \quad (6)$$

and then set

$$\Lambda_{\text{DR}}^2 \times e^{C_{\text{DR}}} = \Lambda_{\text{HE}}^2 \times e^{C_{\text{HE}}} = \Lambda_{\text{PV}}^2 \times e^{C_{\text{PV}}} = \Lambda_{\text{HE}}^2 \times e^{C_{\text{HD}}} \quad (7)$$

for a suitable $O(1)$ numeric constant C_{DR} .

Hint: for the higher-derivative regulator, approximate the modified propagator as

$$\frac{i}{q^2 - m^2 - (q^4/\Lambda^2) + i\epsilon} \approx \frac{i}{q^2 - m^2 + i\epsilon} \times \frac{-\Lambda^2}{q^2 - \Lambda^2 + i\epsilon} \quad (8)$$

where the second factor differs from 1 only for very large momenta. Consequently, for the two propagators in the loop we may further approximate

$$\frac{-\Lambda^2}{q_1^2 - \Lambda^2 + i\epsilon} \approx \frac{-\Lambda^2}{q_2^2 - \Lambda^2 + i\epsilon} \approx \frac{-\Lambda^2}{(q_1 - \xi q_{\text{net}})^2 - \Lambda^2 + i\epsilon}. \quad (9)$$