1. Let's start with the Yukawa theory of a Dirac fermion field Ψ coupled to a real scalar field Φ according to

$$\mathcal{L} \;=\; \overline{\Psi}(i\not\partial - m_f)\Psi \;+\; \tfrac{1}{2}(\partial_\mu\Phi)^2 \;-\; \tfrac{1}{2}m_s^2\Phi^2 \;+\; g\Phi\overline{\Psi}\Psi. \tag{1}$$

For $M_s > 2M_f$, the scalar particle becomes unstable: it decays into a fermion and an antifermion, $S \to f + \bar{f}$.

(a) Calculate the tree-level decay rate $\Gamma(S \to f + \bar{f})$.

(b) In class, we have calculated

$$\Sigma_\Phi^{\text{1 loop}}(p^2) \;=\; \frac{12g^2}{16\pi^2}\int\limits_0^1 d\xi\,\Delta(\xi)\times\left[\frac{1}{\epsilon} \;-\; \gamma_E \;+\; \frac{1}{3} \;+\; \log\frac{4\pi\mu^2}{\Delta(\xi)}\right] \tag{2}$$

$$\text{for } \Delta(\xi) \;=\; m_f^2 \;-\; \xi(1-\xi)p^2. \tag{3}$$

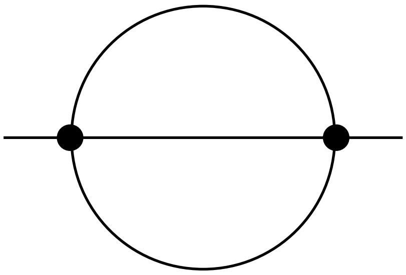
Show that for $p^2 > 4m_f^2$, this $\Sigma_\Phi(p^2)$ has an imaginary part and calculate it for $p^2 = M_s^2 + i\epsilon$.
Note: at this level, you may neglect the difference between m_f^{bare} and M_f^{physical}.

(c) Verify that

$$\operatorname{Im}\Sigma_\Phi^{\text{1 loop}}(p^2 = M_s^2 + i\epsilon) \;=\; -M_s\Gamma^{\text{tree}}(S \to f + \bar{f}) \tag{4}$$

and explain this relation in terms of the optical theorem.

2. And now, a harder exercise about the scalar $\lambda\phi^4$ theory. As discussed in class, in this theory the field strength renormalization begins at the two-loop level. Specifically, the leading contribution to the $d\Sigma(p^2)/dp^2$ — and hence to the $Z - 1$ — comes from the two-loop 1PI diagram



(5)

Your task is to evaluate this contribution.

   (a) First, write the $\Sigma(p^2)$ from the diagram (5) as an integral over two independent loop momenta, say q_1^μ and q_2^μ, then use the Feynman's parameter trick — *cf.* eq. (F.d) of the homework set#13 — to write the product of three propagators as

$$\iiint d\xi\, d\eta\, d\zeta\, \delta(\xi + \eta + \zeta - 1)\, \frac{2}{(\mathcal{D})^3} \tag{6}$$

   where $\mathcal{D}$ is a quadratic polynomial of the momenta q_1, q_2, p, and mass m with Feynman-paramater dependent coefficients.

   Warning: Do not set $p^2 = m^2$ but keep p an independent variable.

   (b) Next, change the independent loop momentum variables from q_1 and q_2 to $k_1 = q_1 +$ something $\times\, q_2 +$ something $\times\, p$ and $k_2 = q_2 +$ something $\times\, p$ to give $\mathcal{D}$ a simpler form

$$\mathcal{D} \;=\; \alpha \times k_1^2 \;+\; \beta \times k_2^2 \;+\; \gamma \times p^2 \;-\; m^2 \;+\; i0 \tag{7}$$

   for some (ξ, η, ζ)–dependent coefficients α, β, γ, for example

$$\alpha \;=\; (\xi + \zeta), \qquad \beta \;=\; \frac{\xi\eta + \xi\zeta + \eta\zeta}{\xi + \zeta}, \qquad \gamma \;=\; \frac{\xi\eta\zeta}{\xi\eta + \xi\zeta + \eta\zeta}. \tag{8}$$

   Make sure the momentum shift has unit Jacobian $\partial(q_1, q_2)/\partial(k_1, k_2) = 1$.

(c) Express the derivative $d\Sigma(p^2)/dp^2$ in terms of

$$\iint d^4k_1\, d^4k_2\, \frac{1}{\mathcal{D}^4}. \tag{9}$$

Note that although this momentum integral diverges as $k_{1,2} \to \infty$, the divergence is logarithmic rather than quadratic.

(d) To evaluate the momentum integral (9), Wick-rotate the momenta k_1 and k_2 to the Euclidean space, and then use the dimensional regularization. Here are some useful formulæ for this calculation:

$$\frac{6}{A^4} = \int_0^\infty dt\, t^3\, e^{-At}, \tag{10}$$

$$\int \frac{d^D k}{(2\pi)^D}\, e^{-ctk^2} = \left(4\pi ct\right)^{-D/2}, \tag{11}$$

$$\Gamma(2\epsilon) X^\epsilon = \frac{1}{2\epsilon} - \gamma_E + \tfrac{1}{2}\log X + O(\epsilon). \tag{12}$$

(e) Assemble your results as

$$\frac{d\Sigma(p^2)}{dp^2} = -\frac{\lambda^2}{12(4\pi)^4} \iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \times \times \left(\frac{1}{\epsilon} - 2\gamma_E + 2\log\frac{4\pi\mu^2}{m^2} + \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta(p^2/m^2))^2}\right). \tag{13}$$

(f) Before you evaluate the Feynman parameter integral (13) — which looks like a frightful mess — make sure it does not introduce its own divergences. That is, without actually calculating the integrals

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3}, \tag{14}$$

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \times \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta(p^2/m^2))^2},$$

make sure that they converge. Pay attentions to the boundaries of the parameter space and especially to the corners where $\xi,\eta \to 0$ while $\zeta \to 1$ (or $\xi,\zeta \to 0$, or $\eta,\zeta \to 0$).

- This calculation shows that

$$\frac{d\Sigma}{dp^2} \;=\; \frac{\text{constant}}{\epsilon} \;+\; \textit{a\_finite\_function}(p^2) \tag{15}$$

and hence

$$\begin{aligned}\Sigma(p^2) \;=\; (\text{a divergent constant}) \;+\; (\text{another divergent constant}) \times p^2 \\ +\; \textit{a\_finite\_function}(p^2)\end{aligned} \tag{16}$$

up to the two-loop order. In fact, this behavior persists to all loops, so all the divergences of $\Sigma(p^2)$ may be canceled with just two counterterms, δ^m and $\delta^Z \times p^2$.

For the purposes of calculating the field strength renormalization factor

$$Z \;=\; \left[1 \;-\; \frac{d\Sigma}{dp^2}\right]^{-1} \tag{17}$$

we need to evaluate the derivative $d\Sigma/dp^2$ at $p^2 = M_{\rm ph}^2$ — the physical mass^2 of the scalar particle. However, to the leading non-trivial order in λ we may approximate $M_{\rm ph}^2 \approx m_{\rm bare}^2$ and set $p^2 = m^2$ in the Feynman-parameter integral (13). Consequently, the second integral (14) becomes a little simpler, although it is still a frigtful mess.

⋆ Optional exercise: Evaluate the integrals (14) for $p^2 = m^2$ and show that

$$\begin{aligned}&\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \;=\; \frac{1}{2}\,, \\ &\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \times \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta)^2} \;=\; -\frac{3}{4}\,.\end{aligned} \tag{18}$$

Do not try to do this calculation by hand — it would take way too much time. Instead, use *Mathematica* or equivalent software. To help it along, replace the (ξ,η,ζ) variables with (x,w) according to

$$\begin{aligned}\xi \;=\; w\times x, \quad \eta \;=\; w\times(1-x), \quad \zeta \;=\; 1-w, \\ \iiint d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \;=\; \int_0^1 dx \int_0^1 dw\, w,\end{aligned} \tag{19}$$

then integrate over w first and over x second.

Alternatively, you may evaluate the integrals like this numerically. In this case, don't bother changing variables, just use a simple 2D grid spanning a triangle defined by $\xi + \eta + \zeta = 1$, $\xi, \eta, \zeta \geq 0$; modern computers can sum up a billion grid points in less than a minute. But watch out for singularities at the corners of the triangle.

(g) Finally, assemble your results and calculate the field strength renormalization factor Z to the two-loop order.