PHY–396 L. Problem set #19. Due March 23, 2021.

1. As explained in class, a coupling g of an operator involving fields $\hat{\phi}_a(x),\ldots,\hat{\phi}_n(x)$ has beta-function

$$\beta_g \;=\; (\gamma_1+\cdots+\gamma_n)\times\bigl(g+\delta^g\bigr)\;-\;\frac{d\delta^g}{d\log E}\,, \tag{1}$$

cf. my notes on the renormalization group, eq. (119) on page 23. When applied to QED, this formula yields

$$\beta_e \;=\; (2\gamma_e+\gamma_\gamma)\times\bigl(e+e\delta_1\bigr)-\;\frac{d(e\delta_1)}{d\log E}\,. \tag{2}$$

Use the Ward identity $\delta_1(E)=\delta_2(E)$ to reduce this formula to

$$\beta_e \;=\; \gamma_\gamma\times e. \tag{3}$$

2. Now consider the electron mass renormalization in QED.

   (a) Calculate to one-loop order the infinite parts of the δ_2 and δm counterterms as functions of the gauge-fixing parameter ξ. Use the off-shell renormalization condition for $E\gg m_e$:

$$\Sigma_{\rm net}(\not{p}) \;=\; A(p^2)\times\not{p}\;+\;B(p^2)\times m;\quad A\;=\;B\;=\;0\quad\text{for}\quad p^2\;=\;-E^2. \tag{4}$$

The counterterms you obtain in part (a) should have form

$$\delta_2(E)\;=\;\frac{C_2(\xi)\alpha}{2\pi}\times\left(\frac{1}{\epsilon}\;+\;\log\frac{\mu^2}{E^2}\;+\;\text{const}\right), \tag{5}$$

$$\delta_m(E)\;=\;\frac{C_m(\xi)\alpha m}{2\pi}\times\left(\frac{1}{\epsilon}\;+\;\log\frac{\mu^2}{E^2}\;+\;\text{const}\right), \tag{6}$$

for some ξ–dependent coefficients $C_2(\xi)$ and $C_m(\xi)$.

   (b) Check that the difference $C_m(\xi)-C_2(\xi)$ does not depend on ξ. If it does, go back to part (a) and check for mistakes.

(c) Show that the dependence of the running electron's mass $m(E)$ on the energy scale E is governed by the equation

$$\frac{dm(E)}{d\log(E)} \;=\; 2\gamma_2 \times \big(m(E)+\delta_m(E)\big) \;-\; \frac{d\delta_m(E)}{d\log(E)}\,. \tag{7}$$

(d) Use eqs. (5) and (6) to show that to the one-loop order

$$\frac{dm(E)}{d\log(E)} \;=\; 2(C_m - C_2) \times \frac{\alpha(E)m(E)}{2\pi} \;+\; O(\alpha^2 m). \tag{8}$$

Note: the running mass should be gauge invariant, that's why I asked you to check the ξ–independence of $C_m - C_2$ in part (b).

(e) Solve the differential equation (8) and show that

$$m(E) \;=\; m_0 \times \left(\frac{\alpha(E)}{\alpha_0}\right)^r \tag{9}$$

for some power r, and calculate that power.
Hint: let $m(E) = F(\alpha(E))$ for some function $F(\alpha)$, then combine eq. (8) and

$$\frac{d\alpha(E)}{d\log E} \;=\; \frac{2\alpha^2}{3\pi} \;+\; O(\alpha^3) \tag{10}$$

into a differential equation for the $F(\alpha)$, and solve that equation.

3. Next, a long reading assignment over the Spring break: *Quantum Mechanics and Path Integrals* by Feynman & Hibbs (find at UT library, read online at scribd.com). Read all you can about care and use of the Path Integrals.

After the spring break, I shall quickly introduce you to the path integrals and then generalize them to the functional integrals of the quantum fields. They are particularly useful for quantizing the gauge fields, especially the non-abelian gauge fields. The path integrals take a while to get used to, so it would help if you become familiar with them before I start to get technical. And that's why I am asking you to read the Feynman & Hibbs book during the break

4. Finally, a few extra reading assignments to the students who need them. These extra assignments are due April 2 (a week and a half after the break), but since this time includes the week of the very hard midterm test, I strongly recommend you read most of this material during the spring break.

   (a) Last semester (Fall 2020 or Fall 2019) I have explained the gauge symmetry — and in particular the non-abelian gauge symmetry — at the semiclassical level. (But I have not calculated any amplitudes in a non-abelian gauge theory, not even at the tree level.) If you have taken QFT 1 with another professor who have not taught this subject, — or if you should have learned it but doubt your knowledge or memory, — read my notes on local symmetries and gauge theories.

   (b) Group theory plays important role in non-abelian gauge symmetries like QCD. I shall try to explain the relevant issues in class, but due to time constraints I would have to be brief, and it would help if you already know the basics. So, if you are unfamiliar with the group theory — especially the continuous group theory, — read *Lie Algebras in Particle Physics: from Isospin to Unified Theories* by Howard Georgi, 1999, Westview press, ISBN 9780813346113 (ebook at UT library). Since you cannot finish the whole book by the time you would need the knowledge, start by *carefully* reading the first 3 chapters, and then browse through chapters on the $SU(2)$, the $SU(3)$, and the color.