1. Let's start with a simple exercise on using path integrals. Consider a 1D particle living on a circle of radius R, or equivalently a 1D particle in a box of length $L = 2\pi R$ with periodic boundary conditions where moving past the $x = L$ point brings you back to $x = 0$. In other words, the particle's position $x(t)$ is defined modulo L.

   The particle has no potential energy, only the non-relativistic kinetic energy $p^2/2M$.

   (a) As a particle moves from some point $x_1 \pmod{L}$ at time t_1 to some other point $x_2 \pmod{L}$ at time t_2, it may travel directly from x_1 to x_2, or it may take a few turns around the circle before ending at the x_2. Show that the space of all such paths on a circle is isomorphic to the space of all paths on an infinite line which begin at fixed x_1 at time t_1 and end at time t_2 at any one of the points $x_2' = x_2 + nL$ where $n = 0, \pm 1, \pm 2, \ldots$ is any whole number.

   Then use path integrals to relate the evolution kernels for the circle and for the infinite line (over the same time interval $t_2 - t_1$) as

$$U_{\text{circle}}(x_2, t_2; x_1, t_1) \;=\; \sum_{n=-\infty}^{+\infty} U_{\text{line}}(x_2 + nL, t_2; x_1, t_1). \tag{1}$$

The next question uses Poisson's resummation formula: If a function $F(n)$ of integer n can be analytically continued to a function $F(\nu)$ of arbitrary real ν, then

$$\sum_{n=-\infty}^{+\infty} F(n) \;=\; \int\! d\nu\, F(\nu) \times \sum_{n=-\infty}^{+\infty} \delta(\nu - n) \;=\; \sum_{\ell=-\infty}^{+\infty} \int\! d\nu\, F(\nu) \times e^{2\pi i \ell \nu}. \tag{2}$$

   (b) The free particle living on an infinite 1D line has evolution kernel

$$U_{\text{line}}(x_2, t_2; x_1, t_1) \;=\; \sqrt{\frac{M}{2\pi i\hbar(t_2 - t_1)}} \times \exp\left(+ \frac{iM(x_2 - x_1)^2}{2\hbar(t_1 - t_1)} \right). \tag{3}$$

   Plug this free kernel into eq. (1) and use Poisson's formula to sum over n.

(c) Verify that the resulting evolution kernel for the particle one the circle agrees with the usual QM formula

$$U_{\text{box}}(x_2,t_2;x_1,t_1) \;=\; \sum_p \psi_p(x)\times\exp\bigl(-i(p^2/2M)(t_2-t_1)/\hbar\bigr)\times\psi_p^*(x_1) \qquad (4)$$

where the momentum p takes circle-quantized values

$$p \;=\; \frac{2\pi\hbar}{L}\times\text{integer} \qquad (5)$$

and

$$\psi_p(x) \;=\; L^{-1/2}\exp(ipx/\hbar) \qquad (6)$$

is the normalized wavefunction of the momentum eigenstate $|p\rangle$.

2. Next, solve the textbook problem 11.1. In this exercise you should learn why spontaneous breakdown of continuous symmetries does not happen in spacetimes of dimensions $d\le 2$. Hint: for a massless free scalar field, the coordinate-space formula for the propagator becomes fairly simple. In d Euclidean dimensions,

$$G_0(x-y) \;\equiv\; \int\frac{d^dp_E}{(2\pi)^d}\,\frac{e^{ip(x-y)}}{p_E^2} \;=\; \frac{\Gamma\bigl(\frac{d}{2}-1\bigr)}{4\pi^{d/2}}\times|x-y|^{2-d}, \qquad (7)$$

except for $d=2$ where $G_0(x-y)=\text{const}-\frac{1}{2\pi}\log|x-y|$.

3. Finally, a modified textbook problem 9.2(c–e) about the Euclidean functional integrals of free quantum fields at finite temperature. Questions (a–d) below concern a free scalar field, questions (e–f) concern free fermionic fields, and question (g) is about the free electromagnetic field.

Note: Although I talked about temperature / coupling correspondence in class, in this exercise there are no couplings — all the fields are free — while the temperature is meant in the usual thermodynamical sense. That is, the fields being in thermal equilibrium with some heat reservoir of temperature T, hence partition function

$$Z = \operatorname{Tr}\bigl(\exp(-\beta\hat{H})\bigr) = \iiiint \mathcal{D}[\text{periodic } \Phi(x_e)] \exp\left(-\int_0^\beta dx^4 \int d^3\mathbf{x}\,\mathcal{L}_E\right). \tag{8}$$

where 'periodic' means periodic in x_4 direction with period $\beta = 1/T$, $\Phi(\mathbf{x}, x_4 + \beta) = \Phi(\mathbf{x}, x_4)$, and likewise for the non-scalar fields.

(a) Consider a free scalar field in $3+1$ dimensions at finite temperature T. Use the Euclidean functional integral (8) to calculate the partition function and hence the Helmholtz free energy $\mathcal{F}(T) = -T\log Z$. Show that formally

$$\mathcal{F}(T) = \frac{T}{2} \times \operatorname{Tr}\log\bigl(-\partial_E^2 + m^2\bigr) \tag{9}$$

where the ∂_E^2 operator acts on functions$(x_1, x_2, x_3, x_4)_E$ which are periodic in the Euclidean time x_4 with period $\beta = 1/T$.

(b) Write down the trace in eq. (9) as a momentum space sum/integral. Then use the Poisson resummation formula (2) to show that

$$\mathcal{F}(T) = \text{const} + \frac{1}{2}\sum_{\ell=-\infty}^{+\infty}\int\frac{d^4p_E}{(2\pi)^4}\exp(i\ell\beta p_4) \times \log(p_E^2 + m^2) \tag{10}$$

$$= \mathcal{F}(0) + \sum_{\ell=1}^{\infty}\int\frac{d^4p_E}{(2\pi)^4}\exp(i\ell\beta p_4) \times \log(p_E^2 + m^2). \tag{11}$$

(c) To evaluate the $\int dp_4$ integral in eq. (11), move the integration contour from the real axis to the two 'banks' of a branch cut. Show that

$$\int_{-\infty}^{+\infty}\frac{dp_4}{2\pi}\exp(i\ell\beta p_4) \times \log(p_4^2 + E^2) = -\frac{\exp(-\ell\beta E)}{\ell\beta}. \tag{12}$$

(d) Finally, use eqs. (11) and (12) to show that the free energy of a free scalar field above the zero-point energy is

$$\mathcal{F}(T) - \mathcal{F}(0) = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,T\,\log\left(1-e^{-\beta E_p}\right) = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\left(\mathcal{F}^{\text{harmonic}}_{\text{oscillator}}(T,E_p) - \tfrac{1}{2}E_p\right). \tag{13}$$

Next, consider a free fermion $0+1$ dimensions, basically a two-level system in Quantum Mechanics. In the Hamiltonian formulation this means

$$\hat{H} = \omega\hat{\psi}^\dagger\hat{\psi} \quad \text{where} \quad \{\hat{\psi},\hat{\psi}^\dagger\} = 1 \quad \text{and} \quad \omega = \text{constant} > 0, \tag{14}$$

while in the Lagrangian formulation, $\psi(t)$ and $\psi^*(t)$ are Grassmann-number-valued functions of the time and

$$L_E = \psi^* \times \frac{d\psi}{dt_e} + \omega \times \psi^*\psi. \tag{15}$$

At finite temperature, all measurable operators must be periodic in Euclidean time with period β, but for the fermionic fields this means that the bilinears must be periodic while the fermionic fields themselves can be either periodic or antiperiodic, $\psi(t_e+\beta) = \pm\psi(t_e)$.

(e) To determine the right choice — periodic or antiperiodic, — use the functional integral to calculate the partition function for both types of boundary conditions for the fermionic variables in the Euclidean time, $\psi(t_E+\beta) = \pm\psi(t_E)$. Show that the periodic condition leads to an unphysical partition function, while the antiperiodic condition leads to the correct partition function of a two-level system.

(f) Now apply the lesson of part (e) to a Dirac fermionic field in $3+1$ dimensions. Calculate the partition function and hence the free energy using the Euclidean path integral over Dirac fields which are antiperiodic in the Euclidean time, $\Psi(\mathbf{x},x_4+\beta) = -\Psi(\mathbf{x},x_4)$.

Finally, consider the free electromagnetic field $A_\mu(x)$. At finite temperature, the $A^\mu(x)$ — just like any other bosonic field — is periodic in the Euclidean time, $A^\mu(\mathbf{x},x_4+\beta) = +A^\mu(\mathbf{x},x_4)$.

(g) Calculate the partition function for the periodic EM field and mind the gauge-fixing terms in the Lagrangian and the Fadde'ev–Popov determinant in the functional integral. Show that formally, the EM free energy is

$$\mathcal{F}(T) \;=\; T \times \operatorname{Tr}\log\bigl(-\partial_E^2\bigr). \tag{16}$$

(h) Recycle arguments from parts (a–d) to show that eq. (16) leads to

$$\mathcal{F}(T) \;-\; \mathcal{F}(0) \;= \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\, 2T \times \left(1 - e^{-\beta|p|}\right). \tag{17}$$