1. This problem continues problems 1 and 2 of the previous homework set. Again, we consider tree-level annihilation of a scalar 'quark' Φ^i and an 'antiquark' Φ^*_j into a pair of gauge bosons with adjoint colors a and b. But this time, we focus on the group theory and on the physical cross-sections rather than the Ward identities.

   (a) Take the annihilation amplitude from part (b) of problem (21.1), focus on its color dependence, and rewrite it in the form

   $$\mathcal{M}(j+i\to a+b) \;=\; F\times\{T^a,T^b\}^i{}_j \;+\; iG\times[T^a,T^b]^i{}_j \tag{1}$$

   where F and G are some functions of all the momenta and of the two vectors' polarizations. Give explicit formulae for F and G.

   (b) Next, let us sum the $|\mathcal{M}|^2$ over the gauge boson's colors and average over the scalars' colors. Show that

   $$\frac{1}{\dim^2(r)}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 \;=\; \frac{C(r)}{\dim(r)}\times\Big(\big(4C(r)-C(\mathrm{adj})\big)\times|F|^2\,+\,C(\mathrm{adj})\times|G|^2\Big). \tag{2}$$

   In particular, for scalars in the fundamental representation of the $SU(N)$ gauge group,

   $$\frac{1}{N^2}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 \;=\; \frac{N^2-1}{2N^2}\left(\frac{N^2-2}{N}\times|F|^2\,+\,N\times|G|^2\right). \tag{3}$$

   (c) Evaluate F and G in the center of mass frame, where the vector particles' polarizations $e^\mu_{1,2}=(0,\mathbf{e}_{1,2})$ are purely spatial and transverse to the vectors' momenta $\pm\mathbf{k}$. For simplicity, use planar rather than circular polarizations.

   (d) Assemble your results and calculate the (polarized, partial) cross-section for the annihilation process.

2. Next, let's evaluate a few one-loop diagrams. In class, I have calculated the (infinite parts of the) δ_2 and δ_1 counterterms for the quarks. Your task is to calculate the analogous $\delta_2^{(\mathrm{gh})}$ and $\delta_1^{(\mathrm{gh})}$ counterterms for the *ghosts fields.*

   (a) Draw one-loop diagrams whose divergences are canceled by the respective counterterms $\delta_2^{(\mathrm{gh})}$ and $\delta_1^{(\mathrm{gh})}$, and calculate the group factors for each diagrams.

   (b) Calculate the momentum integrals for the diagrams. Focus on the UV divergences and ignore the finite parts of the integrals.

   (c) Assemble your results and show that the *difference* $\delta_1^{(\mathrm{gh})} - \delta_2^{(\mathrm{gh})}$ for the ghosts is exactly the same as the $\delta_1 - \delta_2$ difference for the quarks.

3. Finally, consider the three gauge couplings of the $SU(3) \times SU(2) \times U(1)$ Standard Model and their one-loop beta-functions

$$\beta_1^{1\,\mathrm{loop}} = \frac{b_1 g_1^3}{16\pi^2}\,, \quad \beta_2^{1\,\mathrm{loop}} = \frac{b_2 g_2^3}{16\pi^2}\,, \quad \beta_3^{1\,\mathrm{loop}} = \frac{b_3 g_3^3}{16\pi^2}\,. \tag{4}$$

   In this exercise, you do not need to calculate these beta-function from scratch by evaluating the UV divergences of a bunch of loop diagrams. Instead, use eqs. (119) and (121–2) from my notes on QCD beta-function (pages 24–25).

   (a) Calculate the b_1, b_2, b_3 coefficients for the minimal version of the Standard Model: the $SU(3) \times SU(2) \times U(1)$ gauge fields, one Higgs doublet, three families of quarks and leptons, and nothing else.

   ⋆ FYI, each family comprises 8 left-handed Weyl fields in the $(\mathbf{3}, \mathbf{2}, y = +\frac{1}{6})$ and $(\mathbf{1}, \mathbf{2}, y = -\frac{1}{2})$ multiplets of the gauge symmetry and 7 right-handed Weyl fermions in the $(\mathbf{3}, \mathbf{1}, y = +\frac{2}{3})$, $(\mathbf{3}, \mathbf{1}, y = -\frac{1}{3})$, and $(\mathbf{1}, \mathbf{1}, y = -1)$ multiplets.

   (b) Re-calculate the b_1, b_2, b_3 for the MSSM — the Minimal Supersymmetric Standard Model. FYI, here is complete list of the MSSM fields:

   ∘ The $SU(3) \times SU(2) \times U(1)$ gauge fields, same as the non-SUSY SM.

   • For each vector field there is a Majorana fermion (a gaugino) with similar $SU(3) \times SU(2) \times U(1)$ quantum numbers. Altogether, there is an adjoint multiplet of gauginos for each factor of the gauge symmetry.

- ◦ 3 families of quarks and leptons, same as the non-SUSY SM.
- For each Weyl fermion — left-handed or right-handed — in these three families, the MSSM also have a complex scalar field (a squark or a slepton) with similar $SU(3) \times SU(2) \times U(1)$ quantum numbers. Altogether, this makes 45 complex scalar fields in similar multiplets to the quarks and leptons.
- The Higgs sector of the MSSM comprises *two* $SU(2)$ doublets of complex scalars accompanied by one $SU(2)$ doublet of Dirac fermions (the higgsinos); all these doublets have $y = \frac{1}{2}$.

— There are all kinds of Yukawa and ϕ^4 interactions between the MSSM fields, but you do not need them for the one-loop calculation of the gauge couplings' beta-functions.

In Grand Unified Theories

$$\alpha_3 \;=\; \alpha_2 \;=\; \tfrac{5}{3}\alpha_1 \;=\; \alpha_{\text{GUT}} \quad \text{at the GUT scale.} \tag{5}$$

At lower energy scales $E \ll M_{\text{GUT}}$ the SM couplings are given (lo the leading one-loop order) by the Georgi–Quinn–Weinberg equations

$$\begin{aligned}
\frac{1}{\alpha_3(E)} &= \frac{1}{\alpha_{\text{GUT}}} + b_3 \times \frac{1}{2\pi}\log\frac{M_{\text{GUT}}}{E}\,, \\
\frac{1}{\alpha_2(E)} &= \frac{1}{\alpha_{\text{GUT}}} + b_2 \times \frac{1}{2\pi}\log\frac{M_{\text{GUT}}}{E}\,, \\
\frac{1}{\alpha_1(E)} &= \frac{5/3}{\alpha_{\text{GUT}}} + b_1 \times \frac{1}{2\pi}\log\frac{M_{\text{GUT}}}{E}\,.
\end{aligned} \tag{6}$$

(c) Derive these equations from eqs. (4).

(d) The experimental data interpreted in terms of the $\overline{\text{MS}}$ gauge couplings at $E = M_{\text{top}} \approx$ 173 GeV and translated to the $\overline{\text{MS}}$ give

$$\frac{1}{\alpha_3(M_Z)} \approx 9.23, \quad \frac{1}{\alpha_2(M_Z)} \approx 29.97, \quad \frac{1}{\alpha_1(M_Z)} \approx 97.76. \tag{7}$$

Check that these couplings are consistent with eqs. (6) for the MSSM but not for the non-SUSY minimal Standard Model. For the MSSM, calculate the Grand Unification scale M_{GUT} and the unified gauge coupling α_{GUT}.

Although all the additional particles of the MSSM are heavier than M_{top}, for this exercise you should ignore the thresholds due to these masses. Instead, use the b_3, b_2, b_1 coefficients of the massless theory — the minimal SM or the MSSM — for all energies between the M_{top} and the M_{GUT}.