1. Let's start by continuing problem 3 from the previous homework#23. In any even spacetime dimension $d = 2n$, the right hand side of the axial anomaly equation

$$\partial_\mu J^\mu_A \;=\; -\frac{2}{n!}\left(\frac{g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\,\mathrm{tr}\Big(F_{\alpha_1\beta_1}F_{\alpha_2\beta_2}\cdots F_{\alpha_n\beta_n}\Big) \qquad (1)$$

is always a total derivative,

$$\epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\,\mathrm{tr}\Big(F_{\alpha_1\beta_1}\cdots F_{\alpha_n\beta_n}\Big) \;=\; \partial_\mu\Omega^\mu_{(2n-1)} \qquad (2)$$

where $\Omega^\mu_{(2n-1)}$ is some polynomial in gauge fields A^ν and $F^{\rho\sigma}$. For example

$$\begin{aligned} &\text{in } d=2,\ \Omega^\mu_{(1)} \;=\; 2\epsilon^{\mu\nu}\,\mathrm{tr}(A_\nu) \quad [\text{abelian } A_\nu \text{ only}],\\ &\text{in } d=4,\ \Omega^\mu_{(3)} \;=\; 2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(A_\nu F_{\rho\sigma} - \frac{2ig}{3}A_\nu A_\rho A_\sigma\right),\\ &\text{in } d=6,\ \Omega^\mu_{(5)} \;=\; 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\left(A_\nu F_{\rho\sigma}F_{\alpha\beta} - igA_\nu A_\rho A_\sigma F_{\alpha\beta} - \frac{2g^2}{5}A_\nu A_\rho A_\sigma A_\alpha A_\beta\right), \end{aligned} \qquad (3)$$

etc., etc. The $\Omega^\mu_{(2n-1)}$ vectors are equivalent to $(2n-1)$–index totally antisymmetric tensors called the *Chern–Simons forms*, and those forms play many important roles in gauge theory and string theory. In particular, we may use the $\Omega^\mu_{(2n-1)}$ to define a conserved axial current

$$J^\mu_A \;\to\; J^\mu_{AC} \;=\; \overline{\Psi}\Gamma\gamma^\mu\Psi \;+\; \frac{1}{n!}\left(\frac{g}{4\pi}\right)^n \times \Omega^\mu_{(2n-1)}\,. \qquad (4)$$

(Its conservation follows from eqs. (1) and (2).) However, the price of this current conservation is the loss of gauge invariance: the original axial current J^μ_A is gauge invariant, but the J^μ_{AC} is not.

You task is to verify eqs. (2) for $d = 2, 4, 6$.

2. Next, a reading assignment: §19.3 of *Peskin & Schroeder* about the chiral symmetry of QCD and the pions.

For a deeper discussion of pions (and Goldstone bosons in general), please also read chapter 19 of *Weinberg*.

3. The pions are pseudo-Goldstone bosons of the spontaneously broken chiral symmetry of QCD, so they can be created or annihilated by the axial isospin currents

$$J^a_{\mu 5}(x) \;=\; \overline{\Psi}(\bar{u},\bar{d})\gamma^\mu\gamma^5\left(\frac{\tau^a}{2}\right)_{\text{isospin}}\Psi(u,d) = \; -f_\pi\partial_\mu\pi^a(x) \;+\; \text{multi-pion terms.} \tag{5}$$

The f_π in this formula is the *pion decay constant* because it controls the decay rate of the charged pions, mostly into muons and neutrinos, $\pi^+ \to \mu^+\nu_\mu$ and $\pi^- \to \mu^-\bar{\nu}_\mu$. In this exercise, we shall see how this works. Experimentally, $f_\pi \approx 93$ MeV.

The weak interactions at energies $O(M_\pi) \ll M_W$ are governed by the Fermi's current-current effective Lagrangian

$$\mathcal{L}_{\text{Fermi}} \;=\; -2\sqrt{2}G_F J_L^{+\alpha} J_{L\alpha}^{-} \tag{6}$$

where $L_L^{\pm\alpha} = \frac{1}{2}(J_V^{\pm\alpha} - J_A^{\pm\alpha})$ are the left-handed charged currents. In terms of the quark and lepton fields,

$$\begin{aligned} J_L^{+\alpha} \;&=\; \tfrac{1}{2}\overline{\Psi}(\nu_\mu)(1-\gamma^5)\gamma^\alpha\Psi(\mu) \;+\; \cos\theta_c \times \tfrac{1}{2}\overline{\Psi}(u)(1-\gamma^5)\gamma^\alpha\Psi(d) \;+\; \cdots, \\ J_L^{-\alpha} \;&=\; \tfrac{1}{2}\overline{\Psi}(\mu)(1-\gamma^5)\gamma^\alpha\Psi(\nu_\mu) \;+\; \cos\theta_c \times \tfrac{1}{2}\overline{\Psi}(d)(1-\gamma^5)\gamma^\alpha\Psi(u) \;+\; \cdots, \end{aligned} \tag{7}$$

where the $\cdots$ stand for other fermions of the Standard Model, and $\theta_c \approx 13^\circ$ is the Cabibbo angle.

For the pion decay process, the axial part one of the charged currents annihilates the charged pion according to eq. (5) while the other charged current creates the lepton pair.

(a) Show that

$$\langle\text{vacuum}|\,\hat{J}_L^{-\alpha}\,|\pi^+\rangle \;=\; \frac{if_\pi\cos\theta_c}{\sqrt{2}} \times p^\alpha(\pi^+) \tag{8}$$

and therefore the tree-level pion decay amplitude is

$$\mathcal{M} \;=\; \langle\mu^+,\bar{\nu}_\mu|\,\hat{\hat{L}}_{\text{Fermi}}\,|\pi^+\rangle \;=\; iG_f f_\pi\cos\theta_c \times p^\alpha(\pi^+) \times \bar{u}(\nu_\mu)(1-\gamma^5)\gamma_\alpha v(\mu^+). \tag{9}$$

(b) Sum over the fermion spins and calculate the decay rate $\Gamma(\pi^+ \to \mu^+\nu_\mu)$.

(c) Experimentally, $f_\pi \approx 93$ MeV, $M_\pi \approx 140$ MeV, $M_\mu \approx 106$ MeV, $M_\nu \approx 0$, $G_F \approx 1.17 \cdot 10^{-5}\,\mathrm{GeV}^{-2}$, and $\theta_c \approx 13°$. Use these data to calculate the charged pion's lifetime and compare to the experimental value $\tau(\pi^\pm) = 2.6 \times 10^{-8}$ s.

(d) The charged pions decay to muons much more often than they decay to electrons,

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{M_e^2}{M_\mu^2}\frac{(1-(M_e/M_\pi)^2)^2}{(1-(M_\mu/M_\pi)^2)^2} \approx 1.2 \cdot 10^{-4}. \tag{10}$$

Derive this formula, then explain this preference for the heavier final-state lepton in terms of mismatch between lepton's chirality and helicity.

4. Finally, consider the neutral pion decay into two photons, $\pi^0 \to \gamma\gamma$. This decay is facilitated by the QED anomaly of the axial isospin current $J^3_{\mu 5} = -f_\pi \partial_\mu \pi^0 + \cdots$, *cf.* eq. (5). As explained in class,

$$\mathrm{tr}\left(\frac{\tau^3}{2} \times Q^2_{\mathrm{el}}\right) = \frac{e^2}{2} \tag{11}$$

hence

$$\left(\partial^\mu J^3_{\mu 5}\right)_{\text{anomalous}} = -\frac{e^2}{32\pi^2}\,\epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}\,, \tag{12}$$

which may be explained by an effective Lagrangian for the neutral pion field

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2}(\partial_\mu \pi^0)^2 + \frac{e^2}{32\pi^2 f_\pi}\,\pi^0 \times \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}\,. \tag{13}$$

In real life, there is additional contribution to the axial current divergence $\partial^\mu J^3_{\mu 5}$ due to non-zero quark masses; in terms of the effective Lagrangian (13) this extra term can be accounted by the pions mass^2 term, thus

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \pi^0)^2 - \frac{M_\pi^2}{2}(\pi^0)^2 + \frac{e^2}{32\pi^2 f_\pi}\,\pi^0 \times \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}\,. \tag{14}$$

The interaction term here gives rise to the pion decay amplitude

$$\mathcal{M}(\pi^0 \to \gamma\gamma) = -\frac{\alpha}{\pi f_\pi} \times \epsilon^{\alpha\beta\mu\nu}(k_\alpha e^*_\beta)_1 (k_\mu e^*_\nu)_2\,. \tag{15}$$

(a) Derive this amplitude.

(b) Sum $|\mathcal{M}|^2$ over the two photon's polarizations and calculate the neutral pion's decay rate.

(c) Experimentally, $M_\pi \approx 135$ MeV (for the neutral pion), $f_\pi \approx 93$ MeV, and $\alpha \approx 1/137$. Calculate the numerical value of the neutral pion's lifetime for these data and compare to the experimental value of $\tau(\pi^0) \approx 8.5 \times 10^{-17}$ s.