

## Renormalization Scheme Dependence

The running couplings such as  $\lambda(E)$  depend not only on the energy scale  $E$ , but also on the specific rules we use to fix the finite parts of the  $\delta^\lambda(E)$  and other counterterms. If we change those rules — collectively known as the renormalization scheme — then for the same energy scale  $E$  we would get a slightly different running coupling  $\lambda'(E) \neq \lambda(E)$ . The difference is due to quantum corrections which usually start at one loop, thus

$$\lambda'(E) = \lambda(E) + O(\lambda^2(E)). \quad (1)$$

Corollary to this scheme-dependence of the running coupling  $\lambda(E)$ , the beta-function

$$\beta(\lambda) \equiv \frac{d\lambda(E)}{d \log E} \quad (2)$$

also depends on the renormalization scheme,  $\beta'(\lambda') \neq \beta(\lambda)$ . However, this dependence starts at the three-loop level; the one-loop and two-loop terms in the beta-function are the same in all renormalization schemes!

Before I prove this statement, let me make it precise. In a theory with a single running coupling  $\lambda(E)$  (or  $\alpha(E) = e^2(E)/4\pi$ , or  $g^2/4\pi$ , or whatever), the beta-function is a power series

$$\beta(\lambda) = b_1\lambda^2 + b_2\lambda^3 + b_3\lambda^4 + \dots \quad (3)$$

with some constant coefficients  $b_1, b_2, b_3, \dots$ ; each  $b_\ell$  arises at the  $\ell$ -loop order of the perturbation theory. Now let's change the renormalization scheme (for the same theory) so the coupling becomes  $\lambda'(E)$  and the beta-function becomes

$$\beta'(\lambda') = b'_1\lambda'^2 + b'_2\lambda'^3 + b'_3\lambda'^4 + \dots \quad (4)$$

— a power series similar to (3), but maybe with different coefficients  $b'_1, b'_2, b'_3, \dots$

**Theorem:** *the one-loop and two-loop coefficients are the same in all renormalization schemes,  $b'_1 = b_1$  and  $b'_2 = b_2$ , but the three-loop and higher-loop coefficients are scheme dependent,  $b'_\ell \neq b_\ell$  for  $\ell \geq 3$ .*

**Proof:** Let's spell out the relation (1) between the couplings  $\lambda$  and  $\lambda'$  as a power series

$$\lambda'(E) = \lambda(E) + C_1 \times \lambda^2(E) + C_2 \times \lambda^3 + \dots \quad (5)$$

with some constant coefficients  $C_1, C_2, \dots$ . Now consider the inverse couplings  $1/\lambda(E)$  and

$$\frac{1}{\lambda'(E)} = \frac{1}{\lambda(E)} - C_1 + (C_1^2 - C_2) \times \lambda(E) + (-C_1^3 + 2C_1C_2 - C_3) \times \lambda^2(E) + \dots \quad (6)$$

These inverse couplings depend on energy according to

$$\begin{aligned} \frac{d}{d \log E} \frac{1}{\lambda(E)} &= \frac{-1}{\lambda^2(E)} \times \left( \frac{d\lambda(E)}{d \log E} = \beta(\lambda(E)) \right) \\ &= -b_1 - b_2 \times \lambda(E) - b_3 \times \lambda^2(E) - \dots, \end{aligned} \quad (7)$$

and similarly

$$\frac{d}{d \log E} \frac{1}{\lambda'(E)} = -b'_1 - b'_2 \times \lambda'(E) - b'_3 \times \lambda'^2(E) - \dots \quad (8)$$

On the other hand, differentiating both sides of eq. (6) gives us

$$\begin{aligned} \frac{d}{d \log E} \frac{1}{\lambda'(E)} &= \frac{d(1/\lambda')}{d\lambda} \times \left( \frac{d\lambda}{d \log E} = \beta(\lambda) \right) \\ &= \left( \begin{aligned} &\frac{-1}{\lambda^2(E)} - C_1 \times \frac{0}{\lambda(E)} + (C_1^2 - C_2) \times 1 \\ &+ (-C_1^3 + 2C_1C_2 - C_3) \times 2\lambda(E) + \dots \end{aligned} \right) \times \quad (9) \\ &\quad \times \left( b_1 \times \lambda^2(E) + b_2 \times \lambda^3(E) + b_3 \times \lambda^4(E) + \dots \right) \\ &= -b_1 - b_2 \times \lambda(E) - [b_3 - b_1(C_1^2 - C_2)] \times \lambda^2(E) - \dots \\ &= -b_1 - b_2 \times \lambda'(E) - [b_3 - b_2C_1 - b_1(C_1^2 - C_2)] \times \lambda'^2(E) - \dots \end{aligned}$$

Comparing this formula to eq. (8) we immediately see that  $b'_1 = b_1$  and  $b'_2 = b_2$  but  $b'_3 = b_3 - b_2C_1 - b_1(C_1^2 - C_2) \neq b_3$ , and it's obvious that the higher-order coefficients are also renormalization scheme dependent,  $b'_4 \neq b_4$ , etc. *Quod erat demonstrandum.*

A similar theorem applies to theories with multiple couplings  $g_i(E)$ : Write all beta-functions  $\beta_i$  as power series in  $g_1(E), \dots, g_n(E)$  with some numerical coefficients; the coefficients of all terms which arise at the one-loop or two-loop orders do not depend on the choice of the renormalization scheme, but the coefficients of the three-loop and higher-order terms are scheme-dependent.

## Minimal Subtraction

Over the years, field theorists invented all kinds of renormalization schemes. But since 1970's, the most popular schemes are the *Minimal Subtraction* (MS) and its close cousins  $\overline{\text{MS}}$ , DR, and  $\overline{\text{DR}}$ . Here are the rules for the MS scheme:

1. Use dimensional regularization to control the UV divergences.

Note: this rule is peculiar to the Minimal Subtraction and similar schemes. The other renormalization schemes do not care what the UV regulator is, you can use whatever regulator you like as long as it works (*i.e.*, regulates all the UV divergences and does not break symmetries that lead to Ward identities).

2. Identify the  $\mu$  parameter of dimensional regularization

$$\frac{d^4 p}{(2\pi)^4} \rightarrow \mu^{4-D} \times \frac{d^D p}{(2\pi)^D} \quad (10)$$

with the energy scale  $E$  of the renormalization group. This identification sets the ubiquitous logarithms  $\log(\mu^2/E^2)$  to zero.

3. In general, the overall UV divergence of some  $L$ -loop amplitude is a degree- $L$  polynomial in  $1/\epsilon$ , for example

$$\text{---} \rightarrow \text{---} \circlearrowleft \text{---} \supset p \times g^{2L} \times \left( \frac{A_L}{\epsilon^L} + \frac{A_{L-1}}{\epsilon^{L-1}} + \dots + \frac{A_1}{\epsilon} + \text{a finite function of } (p^2) \right) \quad (11)$$

for some constants  $A_L, \dots, A_1$ , and to cancel such a divergence we need an  $L$ -loop-order counterterm

$$\delta_{L \text{ loops}}^Z = g^{2L} \times \left( \frac{A_L}{\epsilon^L} + \frac{A_{L-1}}{\epsilon^{L-1}} + \dots + \frac{A_1}{\epsilon} + A_0 \right). \quad (12)$$

In this counterterm, the coefficients  $A_L, \dots, A_1$  are completely determined by the UV

divergence of the  $L$ -loop diagrams, but the finite free term  $A_0$  is not: its value follows not from the divergence but from the renormalization scheme we use for the amplitude (11).

In the MS scheme, we do not impose any conditions on amplitudes. Instead, we simply set  $A_0 = 0$ . Likewise, the finite parts of all the other counterterms are set to zero.

This is called the *Minimal Subtraction* because all the counterterms do is to subtract the pole at  $\epsilon = 0$ ; the finite part of a divergent amplitude is whatever the loop diagrams produce, the counterterms do not mess with it.

Instead of the original Minimal Subtraction renormalization scheme (MS), people often use the Modified Minimal Subtraction scheme ( $\overline{\text{MS}}$ , pronounced MS-bar). In this scheme, the  $L$ -loop counterterms are degree- $L$  polynomials — without the free term — in

$$\frac{1}{\bar{\epsilon}} \stackrel{\text{def}}{=} \frac{1}{\epsilon} - \gamma_E + \log(4\pi) \quad (13)$$

instead of  $1/\epsilon$ . For example

$$\delta_{L\text{loops}}^Z = g^{2L} \times \left( \frac{A_L}{\bar{\epsilon}^L} + \frac{A_{L-1}}{\bar{\epsilon}^{L-1}} + \dots + \frac{A_1}{\bar{\epsilon}} + 0 \right). \quad (14)$$

This modification makes the regularized net amplitudes somewhat simpler because it subtracts the numerical constants that usually accompany the  $1/\epsilon$  poles.

There are also DR and  $\overline{\text{DR}}$  regularization schemes which are often used in supersymmetric theories. These schemes work similarly to the MS and  $\overline{\text{MS}}$  but use a different ‘flavor’ of dimensional regularization called *dimensional reduction*: all momenta live in  $D = 4 - 2\epsilon$  dimensions, but the vector fields keep all 4 components. Physically, such a reduced 4D vector field comprises one species of a  $D$ -dimensional vector plus  $2\epsilon$  species of scalars with the same mass and charge. Unlike the original ’t Hooft’s dimensional regularization, the dimensional reduction does not break the supersymmetry; apart from that, the difference is usually unimportant.

## Residues, Recursion Relations, and Beta-Functions

In the MS or similar renormalization schemes, the  $L$ -loop counterterms generally comprise poles in  $1/\epsilon$  (or  $1/\bar{\epsilon}$ ) of orders 1 through  $L$ . However, there are recursion relations for all the higher poles  $1/\epsilon^2$ ,  $1/\epsilon^3$ , *etc.*, in terms of the lower-degree poles of the lower-loop-order counterterms. For example, the  $1/\epsilon^2$  pole of a 2-loop counterterm can be obtained from the  $1/\epsilon$  poles of the 1-loop counterterms without doing any 2-loop calculations. Only the simple  $1/\epsilon$  poles have to be calculated the hard way for each loop order: QFT is hard, but not quite as hard as it could be.

Later in these notes, I'll derive the recursion relations for the coefficients of the  $1/\epsilon^2$ , *etc.*, poles for the counterterms combinations used for calculating the beta-functions. There are similar recursion relations for the other counterterm combinations, but I leave them as optional exercise for the students. (In case you have nothing better to do during the summer break, or if you need them for your own research.) But first, let me write down formulae for the beta-function themselves in terms of the residues of the  $1/\epsilon$  poles only.

Let me start with the  $\lambda\phi^4$  theory. Let's define

$$h(\lambda) = \text{coefficient of the } \frac{1}{\epsilon} \text{ pole of } [\delta^\lambda - 2\lambda\delta^Z]. \quad (15)$$

That is, at each loop order, write the  $\delta^\lambda - 2\lambda\delta^Z$  counterterm combination as a polynomial in  $1/\epsilon$ , take the *residue* — the coefficient of the simple  $1/\epsilon$  pole regardless of the higher  $1/\epsilon^2$  through  $1/\epsilon^L$  poles, — and sum over the loop orders. (In theory, over all  $L$  from 1 to  $\infty$ , but in practice one stops at some finite order.) Then in any dimension  $D$

$$\beta(\lambda; D) = (D - 4) \times \lambda + 2\hat{L}h(\lambda) \quad (16)$$

— and in particular, in  $D = 4$  dimensions  $\beta(\lambda) = 2\hat{L}h(\lambda)$  — where

$$\hat{L} = \lambda \frac{d}{d\lambda} - 1 \quad (17)$$

is the operator multiplying an  $L$ -loop term in  $h(\lambda)$  by  $L$ .

In a more explicit form, in the MS scheme the  $L$ -loop-order counterterm combination  $\delta^\lambda - 2\lambda\delta^Z$  has general form

$$\delta_{L\text{loops}}^\lambda - 2\lambda\delta_{L\text{loops}}^Z = \lambda^{L+1} \left( \frac{C_{L,1}}{\epsilon} + \dots + \frac{C_{L,L}}{\epsilon^L} \right) \quad (18)$$

for some numeric constants  $C_{L,1}, \dots, C_{L,L}$ . To obtain the beta-function, we disregard all the higher-pole coefficients  $C_{L,2}, \dots, C_{L,L}$  and focus on the *residue*  $C_{L,1}$  of the simple  $1/\epsilon$  pole. Summing over the loop orders, we have

$$h(\lambda) = C_{1,1}\lambda^2 + C_{2,1}\lambda^3 + C_{3,1}\lambda^4 + \dots \quad (19)$$

and hence in  $D = 4$  dimensions

$$\beta(\lambda; D = 4) = 2C_{1,1}\lambda^2 + 4C_{2,1}\lambda^3 + 6C_{3,1}\lambda^4 + \dots \quad (20)$$

Now consider a QFT with several couplings  $g_1, \dots, g_N$ , where each  $g_s$  is a coefficient of a Lagrangian operator  $\hat{O}_s$ . Each operator has *canonical* energy dimension  $D - \Delta_s(D)$ , so the corresponding coupling has energy dimension  $E^{\Delta_s}$  and its dimensionless strength at energy scale  $\mu$  is

$$\hat{g}_s = g_s \times \mu^{-\Delta_s}. \quad (21)$$

Let's define for each dimensionless coupling  $g_s$  a function

$$h_s(\hat{g}_1, \dots, \hat{g}_N) = \text{coefficient of the } \frac{1}{\epsilon} \text{ pole of } \left[ \delta_s^g - \frac{g_s}{2} \times \sum_i^{\text{fields} \in \hat{O}_s} n_s(i) \delta_i^Z \right] \quad (22)$$

where  $n_s(i)$  is the power of the field $_i$  in the operator  $\hat{O}_s$ . (For example, in the  $\lambda\phi^4$  theory,  $n_\lambda(\phi) = 4$ .) Then in any spacetime dimension  $D$ ,

$$\frac{d\hat{g}_s}{d \log \mu} = \hat{\beta}_s(\hat{g}_1, \dots, \hat{g}_s; D) = -\Delta_s(D) \times \hat{g}_s + 2\hat{L}h_s(g_1, \dots, g_s). \quad (23)$$

For example, consider QED. At high energies, we treat the electron's mass  $m$  as a small coupling between the left-handed and the right-handed electron fields. Thus altogether, we have two couplings: the electric charge  $e(\mu)$  and the mass  $m(\mu)$ , of respective canonical dimensions  $\Delta_e = \frac{1}{2}(4 - D) = \epsilon$  and  $\Delta_m = 1$ . The corresponding  $h$  functions are

$$h_e(\hat{e}, \hat{m}) = \text{Residue}_{(1/\epsilon) \text{ pole}} \left[ e\delta^1 - \frac{e}{2} (2\delta^2 + \delta^3) \right] = \text{Residue}_{(1/\epsilon) \text{ pole}} \left[ -\frac{e}{2} \delta^3 \right] \quad (24)$$

— where the second equality follows from the Ward identity  $\delta^1 = \delta^2$ , — and

$$h_m(\hat{e}, \hat{m}) = \text{Residue}_{(1/\epsilon) \text{ pole}} \left[ \delta^m - \frac{m}{2} \times 2\delta^2 \right]. \quad (25)$$

At the one-loop order, the relevant counterterms are

$$\delta_{1\text{loop}}^3 = -\frac{e^2}{12\pi^2} \times \frac{1}{\epsilon}, \quad \delta_{1\text{loop}}^2 = -\xi \times \frac{e^2}{16\pi^2} \times \frac{1}{\epsilon}, \quad \delta_{1\text{loop}}^m = -(3 + \xi) \times \frac{me^2}{16\pi^2} \times \frac{1}{\epsilon}, \quad (26)$$

where  $\xi$  is the gauge-fixing parameter. Consequently,

$$\begin{aligned} h_e &= -\frac{e}{2} \times \left( -\frac{e^2}{12\pi^2} + O(e^4) \right) \\ &= +\frac{e^3}{24\pi^2} + O(e^5), \\ h_m &= -(3 + \xi) \times \frac{me^2}{16\pi^2} + m \times \xi \times \frac{e^2}{16\pi^2} + O(me^4) \\ &= -\frac{3e^2m}{16\pi^2} + O(e^4m), \end{aligned} \quad (27)$$

and therefore in  $D = 4 - 2\epsilon$  dimensions

$$\hat{\beta}_e = -\epsilon \times \hat{e} + 2 \times \frac{\hat{e}^3}{24\pi^2} + O(\hat{e}^5), \quad (28)$$

$$\hat{\beta}_m = -\hat{m} - 2 \times \frac{3\hat{e}^2\hat{m}}{16\pi^2} + O(\hat{m}\hat{e}^4). \quad (29)$$

Note however that these are beta-functions for the dimensional coupling strengths  $\hat{e} = e/\mu^\epsilon$  and  $\hat{m} = m/\mu$ . For the electric charge  $e$  and the electron mass  $m$  themselves we have

$$\frac{de}{d \log \mu} = \beta_e = \frac{e^3}{12\pi^2} + O(e^5), \quad (30)$$

$$\frac{dm}{d \log \mu} = \beta_m = -m \times \frac{6e^2}{16\pi^2} + O(me^4). \quad (31)$$

DERIVING THE BETA-FUNCTIONS AND THE RECURSION RELATIONS.

In this section we prove the formulae (16) and (23) for the beta-functions and also derive the recursion relations for the higher-pole coefficients. For simplicity, let us start with the  $\lambda\phi^4$  theory.

As explained in class, the renormalized coupling  $\lambda(\mu)$  is related to the bare coupling  $\lambda_b$  according to

$$\lambda_b = \frac{\lambda(\mu) + \delta^\lambda(\mu)}{(1 + \delta^Z(\mu))^2} \quad (32)$$

In the MS scheme, the counterterms are given by power series

$$\begin{aligned} \delta^\lambda &= \sum_{L=1}^{\infty} \lambda^{L+1} \times \sum_{k=1}^L \frac{A_{L,k}}{\epsilon^k}, \\ \delta^Z &= \sum_{L=1}^{\infty} \lambda^L \times \sum_{k=1}^L \frac{B_{L,k}}{\epsilon^k} \end{aligned} \quad (33)$$

with some constant coefficients  $A_{L,k}$  and  $B_{L,k}$ . In the perturbative series like (33) we should treat the coupling  $\lambda$  as infinitesimally small, so small that even  $\lambda/\epsilon$  is small despite the eventual  $\epsilon \rightarrow 0$  limit. In other words, we should take the  $\lambda \rightarrow 0$  limit *before* taking  $\epsilon$  to zero. In this limit, the right hand side of eq. (32) also becomes a power series

$$\frac{\lambda(\mu) + \delta^\lambda(\mu)}{(1 + \delta^Z(\mu))^2} = \lambda(\mu) + \sum_{L=1}^{\infty} \lambda^{L+1}(\mu) \times \sum_{k=1}^L \frac{C_{L,k}}{\epsilon^k} \quad (34)$$

where the coefficients  $C_{L,k}$  are given by polynomials in  $A_{L',k'}$  and  $B_{L',k'}$  with  $L' \leq L$  and  $k' \leq k$ ,

$$C_{1,1} = A_{1,1} - 2B_{1,1}, \quad C_{2,1} = A_{2,1} - 2B_{1,1}, \quad C_{2,2} = A_{22} - A_{11}B_{11} + 3B_{11}^2 - 2B_{2,2}, \dots \quad (35)$$

In particular, for  $k = 1$  and any  $L$ ,

$$C_{L,1} = A_{L,1} - 2B_{L,1}. \quad (36)$$



**Proof:** First, expand the RHS of eq. (32) in powers of the counterterms,

$$\frac{\lambda + \delta^\lambda}{(1 + \delta^Z)^2} = \lambda + (\delta^\lambda - 2\lambda\delta^Z) + (3\lambda(\delta^Z)^2 - 2\delta^\lambda\delta^Z) + O(\delta^3). \quad (37)$$

In the MS scheme, the counterterms do not have finite pieces, so any product of  $N > 1$  counterterms begins with a  $1/\epsilon^N$  divergence, so its expansion in powers of  $\lambda$  and  $1/\epsilon$  contributes only to the  $C_{L,k}$  with  $k \geq N$ . For  $k = 1$ , the  $C_{L,k}$  coefficients come only from expanding the linear  $\delta^\lambda - 2\lambda\delta^Z$  term in eq. (37), hence eq. (36). ■

Now let's re-organize the series (34) by summing over the loop order  $L$  before summing over the pole degree  $k$ , thus

$$\frac{\lambda(\mu) + \delta^\lambda(\mu)}{(1 + \delta^Z(\mu))^2} = \lambda(\mu) + \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k} \quad (38)$$

where

$$f_k(\lambda) \stackrel{\text{def}}{=} \sum_{L=k}^{\infty} C_{L,k} \lambda^{L+1}. \quad (39)$$

In particular, thanks to eq. (36),

$$f_1(\lambda) = \sum_{L=1}^{\infty} (A_{L,1} - 2B_{L,1}) \lambda^{L+1} = \text{Residue}_{(1/\epsilon) \text{ pole}} \left[ \delta^\lambda - 2\lambda\delta^Z \right] \quad (40)$$

— which is exactly the  $h(\lambda)$  function we have earlier defined in eq. (15). In a moment, we shall see that **all the higher-pole coefficients  $f_2(\lambda)$ ,  $f_3(\lambda)$ , etc., are completely determined by the simple-pole coefficient  $f_1(\lambda)$ , and there is a simple formula (16) for the beta-function  $\beta(\lambda)$  in terms of just the  $h(\lambda) = f_1(\lambda)$ .**

The series (38) spells out the right hand side of eq. (32) for any dimension  $D = 4 - 2\epsilon \neq 4$ . Now let's take a closer look at the left hand side for  $D \neq 4$ . The problem with the bare coupling  $\lambda_b$  is that it's dimensionless only in  $D = 4$  dimensions, but in other  $D$  it has dimensionality  $\text{mass}^\Delta$  where

$$\Delta = D - 4 \dim[\Phi] = D - 4 \frac{D-2}{2} = 4 - D = 2\epsilon. \quad (41)$$

The running coupling  $\lambda(\mu)$  suffers from a similar problem, but we can make it dimensionless for any  $D \neq 4$  by rescaling  $\lambda(\mu) \rightarrow [\lambda(\mu)]^{\text{dimensionless}} \times \mu^{2\epsilon}$ . In fact, such rescaling happens

automatically when we include the  $\mu^{2\epsilon}$  factors in the dimensionally regularized momentum integrals (10), so in eq. (38)  $\lambda(\mu)$  is already dimensionless.

But when we apply a similar rescaling to the bare coupling  $\lambda_b$ , we make the coupling dimensionless but  $\mu$ -dependent. In class, we have derived the beta-function from the fact that  $\lambda_b$  was divergent but  $E$ -independent, but now that we work in  $D \neq 4$  dimensions and identify  $E = \mu$ , we should use

$$\lambda_b(\mu) = \frac{\text{divergent constant}}{\mu^{2\epsilon}} \quad (42)$$

on the left hand side of eq. (32). The right hand side of eq. (32) is spelled out in eq. (38); combining all these formulae together, we arrive at

$$\frac{\text{divergent constant}}{\mu^{2\epsilon}} = \lambda_b = \frac{\lambda(\mu) + \delta^\lambda(\mu)}{(1 + \delta^Z(\mu))^2} = \lambda(\mu) + \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k}. \quad (43)$$

Now let's differentiate both sides of eq. (43) with respect to  $\log \mu$ . The right hand side depends on  $\mu$  only via  $\lambda(\mu)$ , hence

$$\frac{d}{d \log \mu} \left( \lambda(\mu) + \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k} \right) = \frac{d\lambda}{d \log \mu} \times \frac{d}{d\lambda}(\dots) = \beta(\lambda(\mu)) \times \left( 1 + \sum_{k=1}^{\infty} \frac{f'_k(\lambda(\mu))}{\epsilon^k} \right) \quad (44)$$

where  $f'_k(\lambda)$  is  $df_k/d\lambda$ . On the left hand side of eq. (43), the  $\mu$ -dependence is explicit but we don't know the constant coefficient. Instead, we may obtain it from the eq. (43) itself, thus

$$\frac{d}{d \log \mu} \left( \frac{\text{const}}{\mu^{2\epsilon}} \right) = -2\epsilon \times \frac{\text{same const}}{\mu^{2\epsilon}} = -2\epsilon \times \left( \lambda(\mu) + \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k} \right) \quad (45)$$

and therefore

$$-2\epsilon\lambda - 2 \sum_{k=1}^{\infty} \frac{f_k(\lambda)}{\epsilon^{k-1}} = \beta(\lambda) \times \left( 1 + \sum_{k=1}^{\infty} \frac{f'_k(\lambda)}{\epsilon^k} \right). \quad (46)$$

At this point, let's treat both sides of eq. (46) as Laurent power series<sup>★</sup> in  $\epsilon$ . On the right hand side, the beta-function  $\beta(\lambda)$  depends on the spacetime dimension, so we should treat it

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★ Unlike the Taylor series which sums up only non-negative powers of some variable, the Laurent series includes both positive and negative powers. A function  $f(z)$  that's singular at  $z = 0$  but is analytic in some ring  $r_1 < |z| < r_2$  in complex  $z$  plane can be expanded into a Laurent series in both positive and negative powers of  $z$ .

as  $\beta(\lambda, \epsilon)$  and expand

$$\beta(\lambda, \epsilon) = \beta_0(\lambda) + \epsilon \times \beta_1(\lambda) + \epsilon^2 \times \beta_2(\lambda) + \dots \quad (47)$$

Note that only non-negative powers of  $\epsilon$  appear in this expansion because the beta-function does not have a singularity at  $D = 4$ . Thus, eq. (43) becomes

$$-2\epsilon\lambda - 2 \sum_{k=1}^{\infty} \frac{f_k(\lambda)}{\epsilon^{k-1}} = \left( \sum_{n=0}^{\infty} \beta_n(\lambda) \times \epsilon^{+n} \right) \times \left( 1 + \sum_{k=1}^{\infty} \frac{f'_k(\lambda)}{\epsilon^k} \right), \quad (48)$$

and the coefficients of similar powers of  $\epsilon$  should be equal on both sides of this equation. In particular, since the left hand side does not contain any powers of  $\epsilon$  greater than +1, the right hand side should not contain them either, and this can happen only if the expansion (47) for the beta-functions stops after the linear term,

$$\beta(\lambda, \epsilon) = \beta_0(\lambda) + \epsilon \times \beta_1(\lambda) + \text{nothing else.} \quad (49)$$

This fact greatly simplifies eq. (48) — it becomes

$$-2\epsilon\lambda - 2f_1 - 2 \sum_{k=2}^{\infty} \frac{f_k}{\epsilon^{k-1}} = \epsilon\beta_1 + \beta_0 + \beta_1 \times f'_1 + \sum_{k=2}^{\infty} \frac{\beta_1 f'_k}{\epsilon^{k-1}} + \sum_{k=1}^{\infty} \frac{\beta_0 f'_k}{\epsilon^k}, \quad (50)$$

and now it's easy to compare similar powers of  $\epsilon$  on both sides. Starting with  $\epsilon^{+1}$  and going down, we have

$$\beta_1(\lambda) = -2\lambda, \quad \text{exactly,} \quad (51)$$

$$\beta_0(\lambda) = -2f_1(\lambda) - \beta_1(\lambda) \times f'_1(\lambda), \quad (52)$$

$$-\beta_1(\lambda) \times f'_2(\lambda) - 2f_2(\lambda) = \beta_0(\lambda) \times f_1(\lambda), \quad (53)$$

.....

$$-\beta_1(\lambda) \times f'_k(\lambda) - 2f_k(\lambda) = \beta_0(\lambda) \times f_{k-1}(\lambda). \quad (54)$$

These formulae give us everything we want to know in terms of the  $f_1(\lambda)$  function, which summarizes the simple poles in the  $\delta^\lambda$  and  $\delta^Z$  counterterms according to eq. (40). In particular, eqs. (51) and (52) give us the beta-function in any spacetime dimension  $D = 4 - 2\epsilon$ ,

$$\beta(\lambda) = (D - 4) \times \lambda + \left( 2\lambda \frac{d}{d\lambda} - 2 \right) f_1(\lambda), \quad (55)$$

— which proves eq. (16).

Moreover, eqs. (53)–(54) give us the recursion relations for the  $f_k(\lambda)$  functions for the higher  $1/\epsilon^k$  poles with  $k \geq 2$ . Indeed, plugging in  $\beta_1 = -2\lambda$  into eq. (54) leads to

$$\forall k \geq 2: \quad \left( 2\lambda \frac{d}{d\lambda} - 2 \right) f_k(\lambda) = \beta_0(\lambda) \times \frac{df_{k-1}(\lambda)}{d\lambda} \quad (56)$$

These differential equations are subject to initial conditions

$$f_k = O(\lambda^{k+1}) \quad \text{for } \lambda \rightarrow 0. \quad (57)$$

Once we know the  $f_1(\lambda) = h(\lambda)$  and hence the beta-function  $\beta_0(\lambda)$ , we may easily solve eqs. (56) and (57) for the  $f_2(\lambda)$ , then solve similar equations for the  $f_3(\lambda)$ , then for the  $f_4(\lambda)$ , *etc., etc.*

Thus, all the  $f_k(\lambda)$  functions for  $k \geq 2$  are completely determined by the  $f_1(\lambda)$  function. Or in terms of the double series

$$\frac{\lambda(\mu) + \delta^\lambda(\mu)}{(1 + \delta^Z(\mu))^2} = \lambda(\mu) + \sum_{L=1}^{\infty} \lambda^{L+1}(\mu) \times \sum_{k=1}^L \frac{C_{L,k}}{\epsilon^k}, \quad (34)$$

the  $C_{L,1}$  coefficients of the simple  $1/\epsilon$  pole at each loop order  $L$  completely determine all the higher poles  $1/\epsilon^2$ ,  $1/\epsilon^3$ , *etc.*

★ ★ ★

Now consider a generic QFT with several couplings  $g_s(\mu)$ ,  $s = 1, \dots, n$ . Similar to  $\lambda(\mu)$ , we make all  $g_s(\mu)$  dimensionless by multiplying them by appropriate powers of  $\mu$ . Then for each coupling we have an equation similar to eq. (43):

$$g_{s,\text{bare}} = \frac{\text{const}}{\mu^{\Delta_s}} = \frac{g_s(\mu) + \delta_s^g(\mu)}{\prod_i^{\text{fields} \in \hat{O}_s} [1 + \delta_i^Z(\mu)]^{n_s(i)/2}} = g_s(\mu) + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} f_s^{(k)}(g_1(\mu), \dots, g_n(\mu)). \quad (58)$$

On the right hand side here, the  $f_1(g_1, \dots, g_n)$  function follows from the simple  $1/\epsilon$  poles in the counterterms, specifically

$$f_s^{(1)}(g_1, \dots, g_n) = \text{Residue}_{(1/\epsilon) \text{ pole}} \left[ \delta_s^g - \frac{g_s}{2} \times \sum_i^{\text{fields} \in \hat{O}_s} n_s(i) \times \delta_i^Z \right] \quad (59)$$

— which is exactly the  $h_s(g_1, \dots, g_n)$  function we have defined earlier in eq. (22), — while the other  $f_s^{(k>1)}$  follow from the higher-order poles in the counterterms. The specific formulae for

the  $f_s^{(2)}$ , *etc.*, in terms of those higher poles are rather complicated, but fortunately we do not need them to calculate the beta functions.

On the left hand side of eq. (58),  $\Delta_s$  is the canonical dimensionality of the bare coupling  $g_s$  in  $D = 4 - 2\epsilon$  dimensions. In general, different couplings have different dimensionalities, but fortunately they are always linear functions of spacetime dimension  $D$  and hence of the  $\epsilon$ ,

$$\Delta_s(\epsilon) = \Delta_s^{(0)} + K_s \times \epsilon, \quad \text{exactly.} \quad (60)$$

For the renormalizable couplings  $\Delta_s^{(0)} = 0$  while  $K_s = \text{valence}(\text{vertex}) - 2$ : for the gauge and Yukawa couplings  $K_s = 3 - 2 = 1$  while for the 4-scalar coupling  $K_s = 2$ .

Similar to eq. (49), linear dependence of the dimensionalities  $\Delta_s(\epsilon)$  on  $\epsilon$  makes all the beta-functions  $\beta(g_a, \dots, g_n; \epsilon)$  exactly linear with respect to epsilon, which helps us to calculate them in terms of the  $f_s^{(1)}$  functions. Indeed, taking the derivatives of both sides of eq. (58) with respect to the  $\log \mu$ , we obtain

$$\begin{aligned} -(\Delta_s^{(0)} + \epsilon K_s) \times \left( g_s + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} \times f_s^{(k)}(g_1, \dots, g_n) \right) = & \quad (61) \\ = \sum_{p=1}^n \beta_p(g_1, \dots, g_n; \epsilon) \times \left[ \delta_{p,s} + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} \times \frac{\partial f_s^{(k)}(g_1, \dots, g_n)}{\partial g_p} \right]. \end{aligned}$$

And now we treat both sides as Laurent series in powers of  $\epsilon$  and compare coefficients of similar powers on both sides. Since the left hand side does not include any powers greater than  $\epsilon^{+1}$ , the right hand side should not have them either, thus

$$\beta_s(g_1, \dots, g_n; \epsilon) = \beta_s^{(0)}(g_1, \dots, g_n) + \epsilon \times \beta_s^{(1)}(g_1, \dots, g_n) + \text{nothing else,} \quad (62)$$

Consequently, matching powers of  $\epsilon^{+1}, \epsilon^0, \epsilon^{-1}, \dots$ , we obtain

$$-K_s \times g_s = \beta_s^{(1)}, \quad (63)$$

$$-\Delta_s^{(0)} \times g_s - K_s \times f_s^{(1)} = \beta_s^{(0)} + \sum_p \beta_p^{(1)} \times \frac{\partial f_s^{(1)}}{\partial g_p}, \quad (64)$$

$$-\Delta_s^{(0)} \times f_s^{(1)} - K_s \times f_s^{(2)} = \sum_p \left[ \beta_p^{(1)} \times \frac{\partial f_s^{(2)}}{\partial g_p} + \beta_p^{(0)} \times \frac{\partial f_s^{(1)}}{\partial g_p} \right], \quad (65)$$

$$\dots\dots\dots$$

$$-\Delta_s^{(0)} \times f_s^{(k)} - K_s \times f_s^{(k+1)} = \sum_p \left[ \beta_p^{(1)} \times \frac{\partial f_s^{(k+1)}}{\partial g_p} + \beta_p^{(0)} \times \frac{\partial f_s^{(k)}}{\partial g_p} \right]. \quad (66)$$

$$\dots\dots\dots$$

The first two equations here — (63) and (64) — give us the exact formulae for all the  $\beta$  functions in terms of the

$$f_s^{(1)}(g_1, \dots, g_n) \equiv h_s(g_1, \dots, g_n) = \text{Residue}_{(1/\epsilon) \text{ pole}} \left[ \delta_s^g - \frac{g_s}{2} \times \sum_i^{\text{fields} \in \hat{O}_s} n_s(i) \times \delta_i^Z \right], \quad (59)$$

namely

$$\beta_s(g_1, \dots, g_n; D) = -\Delta_s(D) \times g_s + \left( \sum_{p=1}^n K_p g_p \frac{\partial}{\partial g_p} - K_s \right) f_s^{(1)}(g_1, \dots, g_n). \quad (67)$$

When all the couplings are renormalizable, this formula simplifies to

$$\beta_s(g_1, \dots, g_n; D) = -\Delta_s(D) \times g_s + 2\hat{L} f_s^{(1)}(g_1, \dots, g_n) \quad (23)$$

where  $\hat{L}$  is the operator counting the number of loops giving rise to each term in the  $f_s^{(1)}$ . As promised, in the MS regularization scheme, the simple  $1/\epsilon$  poles of the appropriate counterterms completely determine all the beta-functions of the theory. As to the coefficients  $f_s^{(k>1)}$  of the higher-order poles  $1/\epsilon^k$  in the same counterterms, they follow from the simple pole coefficients via the recursion relations (66), or in a more compact form,

$$\left( \sum_{p=1}^n K_p g_p \frac{\partial}{\partial g_p} - K_s \right) f_s^{(k+1)}(g_1, \dots, g_n) = \left( \sum_p \beta_p^{(0)} \times \frac{\partial}{\partial g_p} - \Delta_s^0 \right) f_s^{(k)}(g_1, \dots, g_n). \quad (68)$$

When all the couplings are renormalizable, this formula becomes even simpler:

$$2\hat{L} f_s^{(k+1)}(g_1, \dots, g_n) = \left( \sum_p \beta_p^{(0)}(g_1, \dots, g_n) \times \frac{\partial}{\partial g_p} \right) f_s^{(k)}(g_1, \dots, g_n). \quad (69)$$