Neutrino masses

Back when the Glashow–Weinberg–Salam theory was formulated, the neutrinos were though to be exactly massless. But the later discovery of neutrino oscillations between the ν_e, ν_μ, and ν_τ species calls for tiny but non-zero neutrino masses, $m_\nu < 1$ eV. Or rather, the osculations call for the *neutrino mass matrix* $M^\nu_{\alpha,\beta}$ that is non-diagonal in the weak-interactions basis $(\nu_e, \mu_\mu, \nu_\tau)$. Indeed, consider the effective Hamiltonian for a single ultra-relativistic neutrino particle; in the momentum-species basis,

$$\hat{H} \;=\; \sqrt{\hat{p}^2 + \hat{M}^2} \;\approx\; \hat{p} \;+\; \frac{\hat{M}^2}{2\hat{p}}\,. \tag{1}$$

While a free neutrino flies through distance L from the point where it is produced to the point where it is detected, the second term here causes its species state to oscillate,

$$|p,\alpha\rangle \;\to\; \sum_\beta \exp\left(\frac{iL}{2p}\times M^2\right)_{\alpha,\beta} |p,\beta\rangle \quad \text{(up to an overall phase)}. \tag{2}$$

To illustrate how this works, let me spell out the oscillation matrix for 2 neutrino species, say ν_e and ν_μ. In this case, the 2×2 mass matrix can be written as

$$\begin{aligned} M^2 \;&=\; \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &=\; \frac{m_1^2+m_2^2}{2}\times\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;+\; \frac{m_1^2-m_2^2}{2}\times\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}. \end{aligned} \tag{3}$$

where m_2^1 and m_2^2 are the eigenvalues and θ is the mixing angle between the mass eigenbasis (ν_1, ν_2) and the weak-interaction basis (ν_e, μ_μ). Consequently, the oscillation matrix in eq. (2) becomes (up to an overall phase)

$$\begin{aligned} \exp\left(\frac{iL}{2p}\times M^2\right) \;&=\; \cos\left(\frac{L(m_1^2-m_2^2)}{4p}\right)\times\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad +\; i\sin\left(\frac{L(m_1^2-m_2^2)}{4p}\right)\times\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}. \end{aligned} \tag{4}$$

In particular, the probability of the neutrino changing its species from ν_e to ν_μ (or vice verse) after flying through a long distance L is

$$P(\nu_e \to \nu_\mu) \;=\; \sin^2(2\theta) \times \sin^2\left(\frac{L(m_1^2 - m_2^2)}{4p}\right). \tag{5}$$

For the three neutrino species, the 3×3 oscillation matrix is more complicated, so I am not writing it here. Let me simply say that it depends on the differences $m_1^2 - m_2^2$ and $m_2^2 - m_3^2$ between the mass^2 eigenvalues, and on the CKM-like mixing angles between the weak-interactions basis $(\nu_e, \mu_\mu, \nu_\tau)$ and the mass eigenbasis (ν_1, ν_2, ν_3). Experimentally, the neutrino mixing angles are rather large, much larger than the CKM angles angles for the quarks, while the δm^2 differences are very small. According to the July 2020 best fit from the NuFit website, the neutrino mixing angles are $\theta_{12} = 33.45^\circ \pm 0.75^\circ$, $\theta_{23} = 49.2^\circ \pm 1.2^\circ$, $\theta_{13} = 8.58^\circ \pm 0.13^\circ$, while the CP-violating phase is either $\delta_{13} = 198^\circ \pm 25^\circ$ or $\delta_{13} = 282^\circ \pm 28^\circ$. At the same time, the two independent Δmass^2 differences are $\Delta m_{21}^2 = (7.42 \pm 0.21) \times 10^{-5}$ eV^2 and $\Delta m_{32}^2 = (2.50 \pm 0.03) \times 10^{-3}$ eV^2.

Theoretically, there are two ways to add the neutrino masses to the Glashow–Weinberg–Salam theory. The first possibility is to make the neutrino fields Dirac spinors and give them masses via Yukawa couplings to the Higgs doublet, just like the other fermions of the theory. In terms of the original theory (with massless neutrinos), this means add three right-handed Weyl fields $\psi_R(N^\alpha)$ to the theory, make them $SU(2)$ singlets with zero hypercharges, and give them Yukawa couplings

$$\mathcal{L}_{\text{Yukawa}} \;\subset\; -\sum_{\alpha,\beta} Y^N_{\alpha,\beta} \times \psi^\dagger_R(N^\alpha)\psi^i_L(L^\beta) \times \epsilon_{ij} H^j \;+\; \text{Hermitian conjugates.} \tag{6}$$

When the Higgs scalar gets its VEV

$$\langle H \rangle \;=\; \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad v \approx 247 \text{ GeV}, \tag{7}$$

the Yukawa couplings (6) give rise to the Dirac mass terms for the neutrinos

$$\mathcal{L}_{\text{mass}} \;\supset\; -\sum_{\alpha,\beta} M^N_{\alpha,\beta} \times \psi^\dagger_R(N_\alpha)\psi^i_L(L_\beta) \;+\; H.\ c., \qquad M^N_{\alpha,\beta} \;=\; \frac{v}{\sqrt{2}} \times Y^N_{\alpha,\beta}\,. \tag{8}$$

Similar to the quark mass matrices, the $M^N_{\alpha,\beta}$ matrix can be diagonalized by a suitable change of basis,

$$\psi^i_L(L^\beta) \;\to\; (\tilde{U}^L)^\beta + \;\gamma\psi^i_L(L^\gamma), \quad \psi_R(N^\alpha) \to\; (U^N)^\alpha_{\;\delta}\psi_R(N^\delta), \tag{9}$$

$$M^N \;\to\; (U^N)^* M^N (\tilde{U}^L)^\dagger \;=\; \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \tag{10}$$

and the mismatch between the unitary matrices U^L and $\tilde{U}^L$ respectively diagonalizing the charged leptons' and neutrinos' masses gives rise to a CKM-like mixing matrix $V = \tilde{U}^L (U^L)^\dagger$).

The only problem with this setup is that it does not explain why the neutrinos are so light compared to the other fermions of the Standard Model — million times lighter than even the electron, never mind the heavier leptons or quarks. All we can say is that the Yukawa couplings for the neutrinos are extremely weak, $Y^N \sim 10^{-12}$, but we have no idea why they are so weak.

The other possibility is to make the neutrinos Majorana fermions. In Dirac-spinor notations, a Majorana fermion is a neutral field $\Psi(x) = \gamma^2\Psi^*(x)$. In terms of the Weyl spinor fields,

$$\text{Majorana } \Psi(x) \;=\; \begin{pmatrix} \psi_L(x) \\ -\sigma_2\psi^*_L(x) \end{pmatrix} \quad \text{for the same } \psi_L(x), \tag{11}$$

without an independent $\psi_R(x)$. Thus, a majorana fermion is equivalent to a single Weyl fermion $\psi_L(x)$ together with its conjugate $\psi^\dagger_L(x)$. The free Lagrangian for a Majorana fermion is

$$\mathcal{L} \;=\; \tfrac{1}{2}\overline{\Psi}(i\partial\!\!\!/ - m)\Psi \;=\; i\psi^\dagger_L\bar{\sigma}^\mu\partial_\mu\psi_L \;+\; \frac{m}{2}\,\psi^\top_L\sigma_2\psi_L \;+\; \frac{m}{2}\,\psi^\dagger_L\sigma_2\psi^*_L\,, \tag{12}$$

where the mass term couples the LH Weyl spinor ψ_L to itself rather than to a separate RH Weyl spinor $\psi^\dagger_R$. In a general theory of multiple fermions, mass terms of this type are called the *Majorana masses.*

To give the neutrinos Majorana masses we do not need the independent right-handed neutrino fields $\psi_R(N_\alpha)$. All we need are the left-handed neutrino fields $\psi^1_L(L_\alpha)$ and their conjugates, plus some interactions that would give rise to the Majorana mass terms

$$\mathcal{L}_{\rm mass} \supset \tfrac{1}{2}\sum_{\alpha,\beta} M^\nu_{\alpha,\beta}\big(\psi^1_L(L_\alpha)\big)^\top\sigma_2\psi^1_L(L_\beta) \;+\; \tfrac{1}{2}\sum_{\alpha,\beta} M^{\nu*}_{\alpha,\beta}\big(\psi^1_L(L_\alpha)\big)^\dagger\sigma_2\big(\psi^1_L(L_\beta)\big)^*, \tag{13}$$

Note that the mass matrix in this formula may be complex rather than real, but it should be symmetric $M^\nu_{\beta,\alpha} = M^\nu_{\alpha,\beta}$ because

$$\big(\psi^1_L(L_\beta)\big)^\top\sigma_2\psi^1_L(L_\alpha) \;=\; +\big(\psi^1_L(L_\alpha)\big)^\top\sigma_2\psi^1_L(L_\beta) \tag{14}$$

— the σ_2 matrix is antisymmetric, but the fields are anticommuting fermions.

The neutrino mass terms (13) break the $SU(2)\times U(1)$ gauge symmetry so we cannot put them directly into the Lagrangian of the high-energy theory. Instead, they obtain from the gauge-invariant couplings of the leptons and Higgs fields, which give rise to the mass terms after the Higgs gets its vacuum expectation value. The simplest couplings that will do this job are the Yukawa-like couplings involving two left-handed lepton fields and two Higgs scalars,

$$\mathcal{L}_{LLHH} \;=\; \frac{1}{2}\sum_{\alpha,\beta} C_{\alpha,\beta}\times\big(\epsilon_{ij}H^i\psi^j_L(L_\alpha)\big)^\top\sigma_2\big(\epsilon_{k\ell}H^k\psi^\ell_L(L_\beta)\big) \;+\; H.\,c. \tag{15}$$

Note that the product $\epsilon_{ij}H_i\psi^j_L$ of the Higgs doublet and the left-handed Lepton doublet is $SU(2)$ invariant and has $Y_{\rm net}=0$. This makes $\epsilon_{ij}H_i\psi^j_L$ a gauge-invariant Weyl spinor and allows us to combine two such products into a gauge-invariant, Lorentz-invariant Lagrangian term.

When the Higgs VEV breaks the electroweak gauge symmetry, it also makes neutrino mass terms from the couplings (15). Indeed, substituting Higgs VEV $\langle H\rangle^*_i$ into the interaction terms (15), we obtain the Majorana mass terms for the neutrinos that look exactly like in eq. (13) for

$$M^\nu_{\alpha,\beta} \;=\; \frac{v^2}{2}\times C_{\alpha,\beta}\,. \tag{16}$$

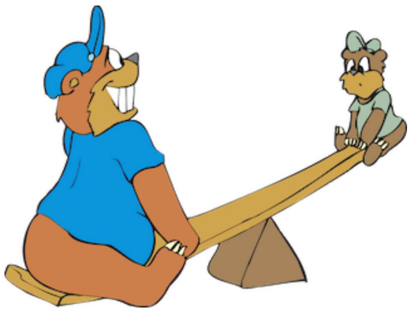
Unlike the dimensionless gauge and Yukawa couplings, the $C_{\alpha,\beta}$ couplings have dimensionality $(\text{energy})^{-1}$. Such couplings make trouble for the perturbation theory at high energies, so they are not allowed in UV-complete quantum field theories. However, if the

Standard Model is only an effective theory that's valid up to some maximal energy $E_{\max}$ but at higher energies must be superseded by a more complete theory, then it's OK for the SM to have *small* negative-dimensionality couplings $C \lesssim (1/E_{\max})$. The key word here is *small* — it explains why the neutrinos are so much lighter than the other fermions: If $C < 1/E_{\max}$, then

$$m_\nu \;\lesssim\; \frac{v^2}{E_{\max}} \;\ll\; v. \tag{17}$$

In particular, for $E_{\max} \sim (10^{15}\,\mathrm{GeV})$ this limit tells us $m_\nu \lesssim 0.1\,\mathrm{eV}$, which is in the right ballpark for the neutrino masses inferred from the neutrino oscillations.

Seesaw Mechanism



The simplest way to generate the $C_{\alpha,\beta}$ couplings in a UV-complete theory is the so-called *seesaw mechanism,* which involves:

1. Right handed neutrino fields $\psi_R(N_\alpha)$ ($\alpha = 1,2,3$) which are completely neutral; in particular, they are $SU(2)$ singlets with $Y = 0$ hypercharge.

2. Majorana mass terms for these neutral RH neutrinos in the Lagrangian,

$$\mathcal{L} \;\supset\; -\frac{1}{2}\sum_{\alpha,\beta} M_N^{\alpha,\beta}\psi_R^\top(N_\alpha)\sigma_2\psi_R(N_\beta) \;+\; \text{Hermitian conjugates}. \tag{18}$$

   These Majorana masses should be very large, $M^N \gg 250$ GeV.

3. Yukawa couplings of the LH and RH neutrinos to the Higgs scalars exactly as in eq. (6), and hence Dirac neutrino masses exactly as in eq. (8) once the Higgs gets its VEV.

Let me explain how the seesaw mechanism works with a 1-family example: 1 LH lepton doublet $L^i = (\nu, e^-)_L$, 1 RH neutrino N, and we ignore all other fields except for the Higgs doublet H^i. The fermionic terms of interest to us are

$$\mathcal{L}_\psi \supset -y\epsilon_{ij}H^i\psi_R^\dagger(N)\psi_L(L^j) - \tfrac{1}{2}M_N\psi_R^\top(N)\sigma_2\psi_R(N) + \text{H. c.}, \tag{19}$$

and once the Higgs gets its VEV (7), we end up with both Dirac and Majorana masses for the neutrinos,

$$\mathcal{L}_{\nu\text{ mass}} = -\frac{yv}{\sqrt{2}}\psi_R^\dagger(N)\psi_L(L^1=\nu) - \tfrac{1}{2}M_N\psi_R^\top(N)\sigma_2\psi_R(N) + \text{H. c.} \tag{20}$$

Trading the RH Weyl spinor $\psi_R(N)$ for an equivalent LH Weyl spinor $\psi_L(\bar{N}) = \sigma_2\psi_R^*(N)$, we get

$$\mathcal{L}_{\nu\text{ mass}} = -\frac{1}{2}\sum_{\alpha,\beta=\nu,\bar{N}}\Big(M_{\alpha\beta}\psi_L^\top(\alpha)\sigma_2\psi_L(\beta) + M^{*\,\alpha\beta}\psi_L^\dagger(\alpha)\sigma_2\psi_L^*(\beta)\Big) \tag{21}$$

where $M_{\alpha\beta}$ is a 2×2 symmetric mass matrix

$$M_\nu = \begin{pmatrix} 0 & m_D \\ m_D & M_N \end{pmatrix} \quad \text{where} \quad m_D = \frac{yv}{\sqrt{2}} \ll M_N\,. \tag{22}$$

Diagonalizing this mass matrix gives us two Majorana mass terms

$$\mathcal{L}_{\nu\text{ mass}} = -\frac{m_1}{2}\psi_L^\top(1)\sigma_2\psi_L(1) - \frac{m_2}{2}\psi_L^\top(2)\sigma_2\psi_L(2) + H.\ c.$$

for eigenvalues

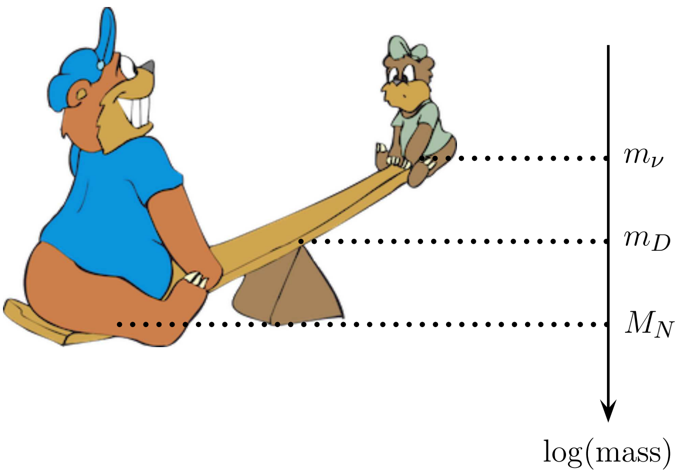
$$m_1 \approx \frac{m_D^2}{M_N} \quad \text{and} \quad m_2 \approx M_N \tag{23}$$

and eigenstates

$$\begin{aligned} \psi_L(1) &\approx \psi_L(\nu) - \frac{m_D}{M_N}\psi_L(\bar{N}) \approx \psi_L(\nu). \\ \psi_L(2) &\approx \psi_L(\bar{N}) + \frac{m_D}{M_N}\psi_L(\nu) \approx \psi_L(\bar{N}), \end{aligned} \tag{24}$$

Qualitatively, the diagonalization leaves the fermionic fields $\psi_L(\nu)$ and $\psi_L(\bar{N})$ almost as they are, but it turns the Dirac mass mixing these fields into a much smaller Majorana mass

$m_\nu = m_D^2/M_N$ for the neutrino ν. Moreover, it is very easy to make this m_ν very small — much smaller than the charged leptons' or quarks' masses — by a simple expedient of starting with very heavy M_N in the Lagrangian. For example, for $m_D = 100$ MeV (similar to the muon mass), raising M_N to 10^8 GeV lowers the m_ν down to 0.1 eV (in the right ball park for the neutrino oscillations). On the logarithmic scale, the extreme mass inequalities $M_\nu \ll m_D \ll M_N$ look like a seesaw

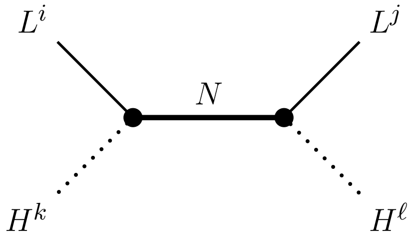


and that's where the name *seesaw mechanism* comes from: with the Dirac neutrino mass acting as the pivot point, the heavier we make the RH neutrino N, the lighter the LH neutrino ν mass comes out.*

Another way to understand the seesaw mechanism is via Feynman diagrams. Consider the amplitude involving two LH leptons and two Higgs scalars; by crossing symmetry, we treat all 4 particles as incoming. This amplitude changes the lepton number by -2, so it must involve the RH neutrino N and its mass term. At the tree level, there are two diagrams

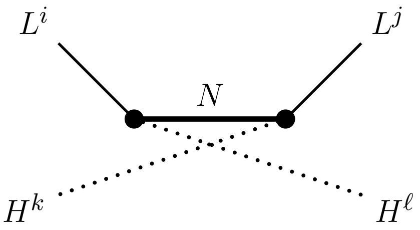
$\star$ By LH vs RH neutrinos I mean their chiralities rather than helicities. In terms of particle helicities, the $\psi(\nu)$ contains the LH neutrinos and the RH antineutrinos, while the $\psi(N)$ contains the RH neutrinos and LH antineutrinos.

related by exchanging the external legs:



$$i\mathcal{M}_1 \;=\; \epsilon^{ik}\epsilon^{j\ell} \times (-iy)^2 \bar{v}\frac{i}{\not{q} - M_N}u, \tag{25}$$

and



$$i\mathcal{M}_2 \;=\; \epsilon^{i\ell}\epsilon^{jk} \times (-iy)^2 \bar{v}\frac{i}{\not{q}' - M_N}u, \tag{26}$$

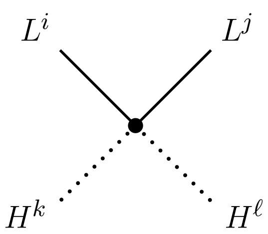
At energies much lower than M_N we may approximate the N propagators in both diagrams as

$$\frac{i}{\not{q} - M_N} \;\approx\; \frac{i}{\not{q}' - M_N} \;\approx\; \frac{-i}{M_N}\,, \tag{27}$$

which in terms of Feynman diagrams means shrinking the N propagator to a point,

$$\mathcal{M}_1 \;+\; \mathcal{M}_2 \;\approx\; \frac{y^2}{M_N} \times \left(\epsilon^{ik}\epsilon^{j\ell} \;+\; \epsilon^{i\ell}\epsilon^{jk}\right). \tag{28}$$

In terms of Feynman diagrams, this approximation corresponds to shrinking the N propagator to a point,



$$i\mathcal{M} \;\approx\; \frac{iy^2}{M_N} \times \left(\epsilon^{ik}\epsilon^{j\ell} \;+\; \epsilon^{i\ell}\epsilon^{jk}\right). \tag{29}$$

In the effective low-energy field theory (for $E \ll M_N$), this diagram stems from the $LLHH$

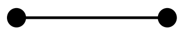
interaction term

$$\mathcal{L} \;=\; \frac{C}{2}\,(\epsilon_{ij}H^i\psi_L^j)^\top\sigma_2(\epsilon_{k\ell}H^k\psi_L^k) \;+\; \text{H. c.} \quad \text{for } C \;=\; \frac{y^2}{M_N}\,, \tag{30}$$

exactly as in eq. (15) (except for only one neutrino flavor here). Consequently, when the Higgs scalar get its VEV, the LH neutrino gets a Majorana mass

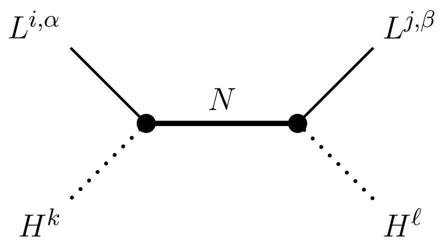
$$m_\nu \;=\; \frac{v^2}{2}\times C \;=\; \frac{v^2y^2}{2M_N} \;=\; \frac{m_D^2}{M_N}\,. \tag{31}$$

Generalization of the seesaw mechanism to 3 LH neutrino species is completely straightforward. Take 3 RH neutrinos with a generic 3×3 mass matrix $M^N_{\alpha\beta}$ and a generic 3×3 matrix $Y^N_{\alpha\beta}$ of Yukawa couplings to the LH neutrinos and the Higgs scalars. Consequently, the Feynman propagators for the N_α field need species indices α,β in addition to the usual Dirac indices (which we suppress)



$$= \left(\frac{i}{\not{q} - M_N + i0}\right)_{\alpha,\beta}\,, \tag{32}$$

the Yukawa vertices also need the species indices, and the Feynman diagrams like



evaluate to

$$i\mathcal{M} \;=\; \epsilon^{ij}\epsilon^{k\ell}\times\sum_{\gamma,\delta}(iY^N_{\alpha\gamma})(iY^N_{\beta\delta})\,\bar{v}\left(\frac{i}{\not{q} - M_N + i0}\right)_{\gamma\delta} u. \tag{33}$$

In the $q \ll M_N$ limit, this amplitude (plus a similar amplitude related by exchanging the scalar legs) evaluates to

$$\mathcal{M}(L^{i,\alpha}, L^{j,\beta}, H^k, H^\ell) \;=\; C_{\alpha,\beta} \times \left(\epsilon^{ik}\epsilon^{j\ell} + \epsilon^{i\ell}\epsilon^{jk}\right) \tag{34}$$

where

$$C_{\alpha,\beta} \;=\; \sum_{\gamma,\delta} Y^N_{\alpha\gamma} Y^N_{\beta\delta} \times \left(\frac{1}{M_N}\right)_{\gamma\delta}\,, \tag{35}$$

or in 3×3 matrix notations,

$$C \;=\; Y_N \times \frac{1}{M_N} \times Y_N^\top. \tag{36}$$

And this is the matrix of $LLHH$ couplings in the effective theory at energies below M_N (but above the electroweak scale of the Higgs VEV). Once the Higgs scalar gets its VEV, these couplings give rise to the Majorana masses of the LH neutrinos; in matrix notations

$$m_\nu \;=\; \frac{v^2}{2} \times C \;=\; m_D \times \frac{1}{M_N} \times m_D^\top. \tag{37}$$

Experimentally, we have no knowledge of the N fields' masses or Yukawa couplings to the LH leptons. All we know is the LH neutrino mass matrix (37), or rather some parameters of this matrix relevant to the neutrino oscillations. Consequently, there is a wide range of Y_N and M_N values consistent with out current knowledge; in particular, M_N could be as low as 1 TeV (for electron-like $y \sim 10^{-6}$) or as high as 10^{15} GeV (for top-quark-like $y \sim 1$). To narrow this range, we would need new Physics beyond the Standard Model.