Phase Space Factors

Consider a quantum transition from some initial state to a continuum of unbound states. For example, an excited atom emitting a photon, or an unstable particle decaying into two or more lighter particles. Another example would be scattering, in which the initial unbound state of two particles about to collide transitions into another unbound state of particles moving in different directions. In all such cases, the final states form a continuum, the transition not to a specific final state but to a continuous family of similar final states. Fermi's golden rule gives the rate of such transitions:

$$\Gamma \stackrel{\text{def}}{=} \frac{d\,\text{probability}}{d\,\text{time}} = \frac{2\pi\rho}{\hbar} \times \left| \langle\text{final}|\,\hat{T}\,|\text{initial}\rangle \right|^2 \tag{1}$$

where $\hat{T} = \hat{H}_{\text{perturbation}} +$ higher order corrections, and ρ is the *density of final states,*

$$\rho = \frac{dN_{\text{final states}}}{dE_{\text{final}}}\,. \tag{2}$$

Equivalently,

$$\Gamma = \int dN_{\text{final}} \left| \langle\text{final}|\,\hat{T}\,|\text{initial}\rangle \right|^2 \times \frac{2\pi}{\hbar}\delta(E_{\text{final}} - E_{\text{initial}}). \tag{3}$$

For an example, consider an atom in an excited state emitting a photon and dropping while the atom itself drops to a lower energy state. For a moment, let's fix the specific initial and finale state of the atom as well as the photon's polarization λ. However, the final states still form a continuous family parametrized by the photon's momentum $\mathbf{p}_\gamma$. In the large-box normalization, the number of such final states is

$$dN_{\text{final}} = \left(\frac{L}{2\pi\hbar}\right)^3 d^3\mathbf{p}_\gamma = \frac{L^3}{(2\pi\hbar)^3} \times p_\gamma^2\,dp_\gamma\,d^2\Omega_\gamma \tag{4}$$

where $d^2\Omega_\gamma$ is the infinitesimal solid angle into which the photon is emitted. At the same time,

$$E^{\rm net}_{\rm final}\ -\ E^{\rm net}_{\rm initial}\ =\ cp_\gamma\ +\ E^{\rm atom}_{\rm final}\ -\ E^{\rm atom}_{\rm initial}\ =\ cp\ -\ \Delta E^{\rm atom}, \tag{5}$$

hence

$$\Gamma\ =\ \frac{1}{(2\pi)^2\hbar^4}\int\!d^2\Omega_\gamma\int\!dp_\gamma\,p_\gamma^2\times L^3\left|\left\langle{\rm atom}_f+\gamma\right|\hat T\left\langle{\rm atom}_i\right|\right|^2\times\delta(cp_\gamma-\Delta E^{\rm atom}) \tag{6}$$

where the L^3 factor cancels against the $L^{-3/2}$ factor in the matrix element due to the photon's wave function in the large-box normalization. Integrating over the p_γ in eq. (6) removes the delta-function for the energy, and we are left with

$$\Gamma\ =\ \frac{(\Delta E^{\rm atom})^2}{(2\pi)^2\hbar^4c^3}\int\!d^2\Omega_\gamma\,L^3\left|\left\langle{\rm atom}_f+\gamma\right|\hat T\left\langle{\rm atom}_i\right|\right|^2. \tag{7}$$

Moreover, we may drop the $\int\!d\Omega$ integral and get the partial rate of photon emission in a particular direction,

$$\frac{d\Gamma}{d\Omega_\gamma}\ =\ \frac{(\Delta E^{\rm atom})^2}{(2\pi)^2\hbar^4c^3}\times L^3\left|\left\langle{\rm atom}_f+\gamma\right|\hat T\left\langle{\rm atom}_i\right|\right|^2. \tag{8}$$

Alternatively, we may not only integrate over the photon's direction but also sum over its polarization as well as some quantum numbers of the atom's final state — such as m_j — that we are not bothering to measure. This gives us a more inclusive transition rate

$$\Gamma\ =\ \frac{(\Delta E^{\rm atom})^2}{(2\pi)^2\hbar^4c^3}\int\!d^2\Omega_\gamma\sum_\lambda\sum_{m_j(f)}L^3\left|\left\langle{\rm atom}_f+\gamma\right|\hat T\left\langle{\rm atom}_i\right|\right|^2. \tag{9}$$

For another example, consider the decay of an unstable particle into n daughter particles. Due to momentum conservation, only $n-1$ of the daughter particle momenta $\mathbf{p}'_i$ are independent, but formally we may integrate over all n of the $\mathbf{p}'_i$ but include a delta-function to reimpose the momentum conservation. Thus,

$$\begin{aligned}\Gamma\ =\ &\int\!\frac{L^3d^3\mathbf{p}'_1}{(2\pi\hbar)^3}\cdots\int\!\frac{L^3d^3\mathbf{p}'_n}{(2\pi\hbar)^3}\left|\left\langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n\right|\hat T\left|\mathbf{p}_{\rm in}\right\rangle\right|^2\times\\ &\times\left(\frac{2\pi\hbar}{L}\right)^3\delta^{(3)}(\mathbf{p}'_1+\cdot+\mathbf{p}'_n-\mathbf{p}_{\rm in})\times\frac{2\pi}{\hbar}\delta(E'_1+\cdots+E'_n-E_{\rm in}).\end{aligned} \tag{10}$$

This formula assumes non-relativistic big-box normalization of quantum states and matrix elements. In high-energy physics we prefer relativistic normalization in which powers of L

go away and we also use $\hbar = 1$ units. On the other hand, the relativistically normalized particle states have extra factors of $\sqrt{2E}$ for each final-state or initial state particle, and these factors must be compensated by dividing the $|\langle\text{matrix element}\rangle|^2$ by $(2E)$ for each initial or final particle. Thus, for a decay of 1 initial particle into n final particles,

$$\Gamma = \frac{1}{2E_{\text{in}}}\int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\cdots\int\frac{d^3\mathbf{p}'_n}{(2\pi)^3\,2E'_n}\left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_{\text{in}}\rangle\right|^2\times(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_{\text{in}}), \tag{11}$$

where the 4D δ function takes care of both momentum conservation and of the denominator dE_f in the density-of-states factor (2). Likewise, the transition rate for a generic $2 \to n$ scattering process is given by

$$\begin{aligned}\Gamma = \frac{1}{2E_1\times 2E_2}\int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\cdots\int\frac{d^3\mathbf{p}'_n}{(2\pi)^3\,2E'_n}\left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_1,p_2\rangle\right|^2\times\\ \times(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_1-p_2).\end{aligned} \tag{12}$$

In terms of the scattering cross-section σ, the rate (12) is $\Gamma = \sigma\times$ flux of initial particles. In the large-box normalization the flux is $L^{-3}|\mathbf{v}_1-\mathbf{v}_2|$, so in the continuum normalization it's simply the relative speed $|\mathbf{v}_1-\mathbf{v}_2|$. Consequently, the total scattering cross-section is given by

$$\begin{aligned}\sigma_{\text{tot}} = \frac{1}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|}\int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\cdots\int\frac{d^3\mathbf{p}'_n}{(2\pi)^3\,2E'_n}\left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_1,p_2\rangle\right|^2\times\\ \times(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_1-p_2).\end{aligned} \tag{13}$$

In particle physics, all the factors in eqs (11) or (13) besides the matrix elements — as well as the integrals over such factors — are collectively called the *phase space* factors.

A note on Lorentz invariance of decay rates or cross-sections. The matrix elements $\langle\text{final}|\,\mathcal{M}\,|\text{initial}\rangle$ are Lorentz invariant, and so are all the integrals over the final-particles' momenta and the δ-functions. The only non-invariant factor in the decay-rate formula (11) is the pre-integral $1/E_{\text{init}}$, hence the decay rate of a moving particle is

$$\Gamma(\text{moving}) = \Gamma(\text{rest frame})\times\frac{M}{E} \tag{14}$$

where M/E is precisely the time dilation factor in the moving frame.

As to the scattering cross-section, it should be invariant under Lorentz boosts along the initial axis of scattering, thus the same cross-section in any frame where $\mathbf{p}_1 \parallel \mathbf{p}_2$. This includes the *lab frame* where one of the two particles is initially at rest, the *center-of-mass frame* where $\mathbf{p}_1 + \mathbf{p}_2 = 0$, and any other frame where the two particles collide head-on. And indeed, in any frame where both $\mathbf{p}_1$ and $\mathbf{p}_2$ are parallel to the z axis, the pre-integral factor in eq. (13) for the cross-section becomes

$$\frac{1}{4E_1E_2|\mathbf{v}_1 - \mathbf{v}_2|} = \frac{1}{4|E_1\mathbf{p}_2 - E_2\mathbf{p}_1|} = \frac{1}{4|\epsilon_{\mu\nu xy}p_1^\mu p_2^\nu|} \tag{15}$$

which is manifestly invariant under the $SO^+(1,1)$ group of Lorentz boosts along the z axis.

Let's simplify eq. (13) for a 2 particle $\to$ 2 particle scattering process in the center-of-mass frame where $\mathbf{p}_1 + \mathbf{p}_2 = 0$. In this frame, the pre-integral factor (15) becomes

$$\frac{1}{4|\mathbf{p}| \times (E_1 + E_2)} \tag{16}$$

while the remaining phase space factors amount to

$$\begin{aligned}
\mathcal{P}_{\text{int}} &= \int\frac{d^3\mathbf{p}_1'}{(2\pi)^3\,2E_1'}\int\frac{d^3\mathbf{p}_2'}{(2\pi)^3\,2E_2'}\,(2\pi)^4\delta^{(3)}(\mathbf{p}_1' + \mathbf{p}_2')\delta(E_1' + E_2' - E_{\text{net}}) \\
&= \int\frac{d^3\mathbf{p}_1'}{(2\pi)^3 \times 2E_1' \times 2E_2'}\,(2\pi)\delta(E_1'(\mathbf{p}_1') + E_2'(-\mathbf{p}_1') - E_{\text{net}}) \\
&= \int d^2\Omega_{\mathbf{p}'} \times \int_0^\infty dp'\,\frac{p'^2}{16\pi^2E_1'E_2'} \times \delta(E_1' + E_2' - E_{\text{tot}}) \\
&= \int d^2\Omega_{\mathbf{p}'}\left[\frac{p'^2}{16\pi^2E_1'E_2'}\Bigg/\frac{d(E_1' + E_2')}{dp'}\right]_{E_1'+E_2'=E_{\text{tot}}}^{\text{when}}.
\end{aligned} \tag{17}$$

On the last 3 lines here $E_1' = E_1'(\mathbf{p}_1') = \sqrt{p'^2 + m_1'^2}$ while $E_2' = E_2'(\mathbf{p}_2' = -\mathbf{p}_1') = \sqrt{p'^2 + m_2'^2}$. Consequently,

$$\frac{dE_1'}{dp'} = \frac{p'}{E_1'}, \quad \frac{dE_2'}{dp'} = \frac{p'}{E_2'}, \tag{18}$$

hence

$$\frac{d(E_1' + E_2')}{dp'} = \frac{p'}{E_1'} + \frac{p'}{E_2'} = \frac{p'}{E_1'E_2'} \times (E_2' + E_1' = E_{\text{tot}}), \tag{19}$$

and therefore

$$\mathcal{P}_{\text{int}} \;=\; \frac{1}{16\pi^2} \times \frac{p'}{E_{\text{tot}}} \times \int d^2\Omega_{\mathbf{p}'}\,. \tag{20}$$

Including the pre-integral factor (16), we arrive at the net phase space factor

$$\mathcal{P} \;=\; \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{tot}}^2} \times \int d^2\Omega_{\mathbf{p}'}\,. \tag{21}$$

The matrix element $\mathcal{M}$ for the scattering should be put inside the direction-angle integral in this phase-space formula. Thus, the total scattering cross-section is

$$\sigma_{\text{tot}}(1+2\to 1'+2') \;=\; \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{cm}}^2} \times \int d^2\Omega\; \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_1+p_2\rangle\right|^2, \tag{22}$$

while the partial cross-section for scattering in a particular direction is

$$\frac{d\sigma(1+2\to 1'+2')}{d\Omega_{\text{cm}}} \;=\; \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{cm}}^2} \times \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_1+p_2\rangle\right|^2. \tag{23}$$

Note: while the total cross-section is the same in all frames where the initial momenta are collinear, but in the partial cross-section, the $d\Omega$ depends on the frame of reference, so eq. (23) applies only in the center-of mass frame. Also, the E_{cm} factor in denominators of both formulae stands for the net energy in the center-of-mass frame. In frame-independent terms,

$$E_{\text{cm}}^2 \;=\; (p_1+p_2)^2 \;=\; (p_1'+p_2')^2 \;=\; s. \tag{24}$$

Finally, let me write down the phase-space factor for a 2-body decay (1 particle $\to$ 2 particles) in the rest frame of the initial particle. The under-the-integral factors for such a decay are the same as in eq. (20) for a $2\to 2$ scattering, but the pre-integral factor is $1/2M_{\text{init}}$ instead of the (16), thus

$$\mathcal{P} \;=\; \frac{p'}{32\pi^2 M^2}\,, \tag{25}$$

meaning

$$\frac{d\Gamma(0\to 1'+2')}{d\Omega} \;=\; \frac{p'}{32\pi^2 M^2} \times \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_0\rangle\right|^2, \tag{26}$$

$$\Gamma(0\to 1'+2') \;=\; \frac{p'}{32\pi^2 M^2} \times \int d^2\Omega\; \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_0\rangle\right|^2. \tag{27}$$