QCD Feynman Rules

The classical chromodynamics has a fairly simple Lagrangian

$$\mathcal{L} \;=\; \mathcal{L}_{\text{Yang−Mills}} \;+\mathcal{L}_{\text{quarks}} \;=\; -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} \;+\; \sum_f \overline{\Psi}_{if}(i\not{D} + m_f)\Psi^{if} \tag{1}$$

where i denotes the color of a quark and f its flavor. In my notations, I follows the usual summation convention for the Lorentz or color indices, — and the Dirac indices are implicit altogether; but the sum over the quark flavors is explicit since the mass m_f depends on the flavor. OOH, the covariant derivatives D_μ are flavor-blind, $D_\mu\Psi^{if} = \partial_\mu\Psi^{if} + igA^a_\mu(t^a)^i{}_j\Psi^{jf}$ where t^a are matrices representing the gauge group generators in the quark representation; in QCD the quarks belong to the fundamental **3** representation of the $SU(3)_C$ so t^a are $\frac{1}{2}\times$ Gell-Mann matrices λ^a.

The Quantum ChromoDynamics is more complicated, even at the Lagrangian level: Besides the physical terms (1), there are gauge-fixing and ghost terms, and then there the counterterms. Altogether, bare QCD Lagrangian is

$$\mathcal{L}_{\text{bare}} \;=\; \mathcal{L}_{\text{ren}} \;+\; \mathcal{L}_{\text{c.t.}}\,, \tag{2}$$

$$\begin{aligned}
\mathcal{L}_{\text{ren}} \;&=\; \mathcal{L}_{\text{phys}} \;+\; \mathcal{L}_{\text{g.f.}} \;+\; \mathcal{L}_{\text{gh}} \\
&=\; -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} \;+\; \sum_f \overline{\Psi}_{if}(i\not{D} + m_f)\Psi^{if} \;-\; \frac{1}{2\xi}\big(\partial_\mu A^{a\mu}\big)^2 +\; \partial_\mu\bar{c}^a D^\mu c^a && (3)\\
&=\; -\frac{1}{2}\big(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\big)^2 \;-\; \frac{1}{2\xi}\big(\partial_\mu A^{a\mu}\big)^2 \;+\; (\partial_\mu\bar{c}^a)(\partial^\mu c^a) \;+\; \sum_f \overline{\Psi}_{if}(i\not{\partial} - m_f)\Psi^{if} && (4)\\
&\quad+\; gf^{abc}A^b_\mu A^c_\nu \partial_\mu A^{a\nu} \;-\; \frac{g^2}{4}(f^{abc}A^b_\mu A^c_\nu)^2 \;-\; gf^{abc}(\partial_\mu\bar{c}^a)A^{b\mu}c^c && (5)\\
&\quad-\; gA^a_\mu \sum_f \overline{\Psi}_{if}\gamma^\mu(t^a)^i{}_j\Psi^{jf}, && (5)
\end{aligned}$$

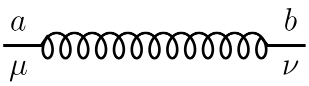
and

$$\begin{aligned}
\mathcal{L}_{\text{c.t.}} \;&=\; -\frac{\delta_3}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 \;+\; g\delta_1^{(3g)}f^{abc}A^b_\mu A^c_\nu\partial_\mu A^{a\nu} \;-\; \frac{g^2\delta_1^{(4g)}}{4}(f^{abc}A^b_\mu A^c_\nu)^2 && (6)\\
&\quad+\; \delta_2^{(\text{gh})}\partial_\mu\bar{c}^a\partial^\mu c^a \;-\; g\delta_1^{(\text{gh})}f^{abc}\partial_\mu\bar{c}^a A^{b\mu}c^c && (6)\\
&\quad+\; \sum_f \overline{\Psi}_{if}\Big(i\delta_2^{(q_f)}\not{\partial} \;+\; \delta_m^{(q_f)} \;-\; g\delta_1^{(q_f)}\not{A}^a t^a\Big)\Psi^{if}. && (6)
\end{aligned}$$

On the last line here, the quark-related counterterms $\delta_2^{(q_f)}$, $\delta_1^{(q_f)}$, and $\delta_m^{(q_f)}$ could be flavor-dependent due to flavor-dependence of the quark mass.

QCD Feynman rules follow from expanding the bare Lagrangian (2) into the renormalized quadratic terms (line (4)), the renormalized cubic and quartic terms (lines (5)), and the counterterms (lines (6)). Specifically, the quadratic terms on line (4) give rise to the gluon, ghost, and quark propagators:

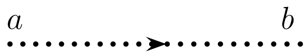
⋈ The first two terms on line (4) are responsible for the gluon propagator



$$= \frac{-i\delta^{ab}}{k^2+i0}\left(g^{\mu\nu} + (\xi-1)\frac{k^\mu k^\nu}{k^2+i0}\right). \tag{7}$$

Apart from the δ^{ab} factor for the adjoint colors, this propagator looks just like the photon propagator in a similar gauge.

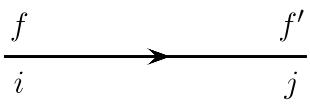
⋈ The third term on the line (4) governs the ghost propagator



$$= \frac{i\delta^{ab}}{k^2+i0}. \tag{8}$$

Note the arrow here, as a ghost c^b is different from an antighost $\bar{c}^a$.

⋈ Finally, the last term on line (4) is responsible for the quark propagator

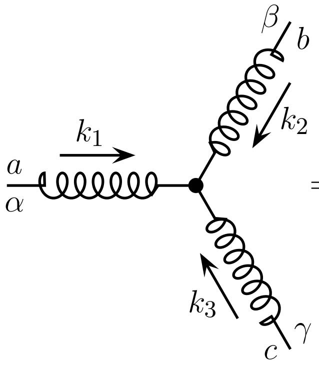


$$= \frac{i\delta^i_j\delta^f_{f'}}{\not{p}-m_f+i0}. \tag{9}$$

Apart from its color and flavor indices — and hence factors δ^i_j and $\delta^f_{f'}$ — and the flavor-dependent mass m_f, the quark propagators looks just like the electron or muon propagator in QED.

Next, consider the cubic and quartic vertices stemming from the interaction terms on lines (5).

- The first term on the top line (5) gives rise to the three-gluon vertex



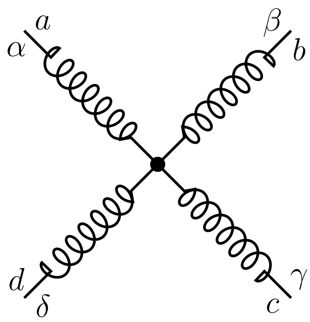
$$= -gf^{abc}\Big[g^{\alpha\beta}(k_1-k_2)^\gamma + g^{\beta\gamma}(k_2-k_3)^\alpha + g^{\gamma\alpha}(k_3-k_1)^\beta\Big]. \tag{10}$$

The specific indexology of this vertex follows from totally symmetrizing the 3 gluon fields A^a_α, A^b_β, and A^c_γ in the interaction term $gf^{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu}$. Since the f^{abc} factor is totally antisymmetric in the adjoint colors of the 3 gluons, this means totally antisymmetrizing the Lorentz and derivative indices, thus

$$\mathcal{L} \supset gf^{abc}(\partial^\gamma A^a_\alpha)A^b_\beta A^c_\gamma g^{\alpha\beta} = \frac{gf^{abc}}{6}\times\begin{pmatrix} g^{\alpha\beta}(\partial^\gamma A^a_\alpha)A^b_\beta A^c_\gamma - g^{\alpha\beta}A^a_\alpha(\partial^\gamma A^b_\beta)A^c_\gamma \\ +g^{\beta\gamma}(\partial^\alpha A^b_\beta)A^c_\gamma A^a_\alpha - g^{\beta\gamma}A^b_\beta(\partial^\alpha A^c_\gamma)A^a_\alpha \\ +g^{\gamma\alpha}(\partial^\beta A^c_\gamma)A^a_\alpha A^b_\beta - g^{\gamma\alpha}A^c_\gamma(\partial^\beta A^a_\alpha)A^b_\beta \end{pmatrix}. \tag{11}$$

After that, in the momentum space each ∂ becomes the appropriate ik and then there is an overall factor of i, hence the 3-gluon vertex (10).

- The second term on the top line (5) is quartic in the gauge field, so it gives rise to the four-gluon vertex



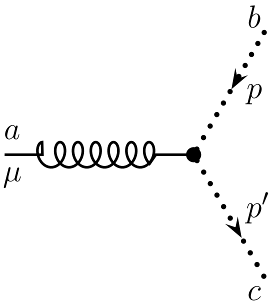
$$= -ig^2\begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}) \\ +f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\gamma\beta}) \\ +f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix}. \tag{12}$$

Again, the messy indexology of this vertex stems from totally symmetrizing the quartic interaction terms WRT to the 4 gauge fields A^a_α, A^b_β, A^c_γ and A^d_δ:

$$\mathcal{L} \supset -\frac{g^2}{4}\left(f^{abe}A^a_\alpha A^b_\beta\right)^2 = -\frac{g^2}{4}\left(f^{abe}A^a_\alpha A^b_\beta\right)g^{\alpha\gamma}g^{\beta\delta}\left(f^{cde}A^c_\gamma A^d_\delta\right)$$
$$= -\frac{g^2}{24}A^a_\alpha A^b_\beta A^c_\gamma A^d_\delta \times \begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix} \tag{13}$$

where $f^{abe}f^{cde}$ is antisymmetric WRT $a \leftrightarrow b$ or $c \leftrightarrow d$ and likewise for the other color-index structures, hence the 4-gluon vertex (12).

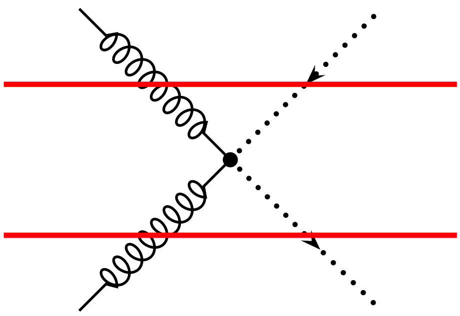
- The third term $-gf^{abc}(\partial^\mu \bar{c}^a)A^b_\mu c^c$ on the top line (5) gives rise to the ghost-antighost-gluon vertex



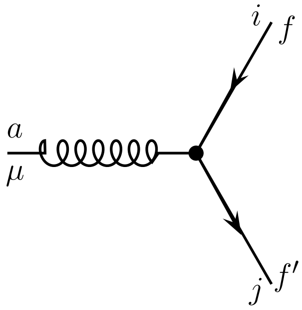
$$= +gf^{abc}p'^\mu. \tag{14}$$

Note that although the ghosts and antighosts look like massless scalar fields, the momentum structure of this vertex is different from a similar vertex for a physical colored scalar field we might add to the theory, of for that matter, from the scalar-photon vertex in scalar QED. For a physical scalar field, the single-gluon vertex involves the sum of two scalar lines' momenta $(p + p')^\mu$, but the ghost vertex (14) involves only the antighost's momentum p'^μ but not the ghost's momentum p^μ.

- ◦ Also, unlike a physical scalar, the ghost fields do not have the $A^\mu A_\mu \bar{c}c$ term in their Lagrangian (color indices suppresses), hence there is no seagull vertex for the ghosts



- Finally, the term on the second line (5) gives rise to the quark-antiquark-gluon vertex similar to the electron-positron-photon vertex in QED,

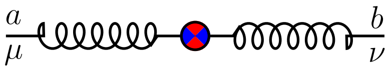


$$= -ig\gamma^\mu \times \delta_f^{f'} \times (t^a)^j_{\ i}\,. \tag{15}$$

The $\delta_f^{f'}$ factor for the quark flavors is similar to not mixing the electron, muon, and tau lepton species in QED, but the color matrix $(t^a)^j_{\ i}$ is new to QCD. I shall return to this matrix later in these notes.

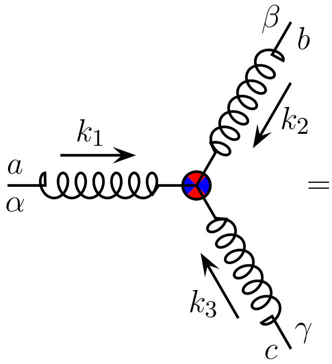
In addition, the renormalized theory has a whole bunch of the counterterm vertices:

∗ The two-gluon counterterm vertex



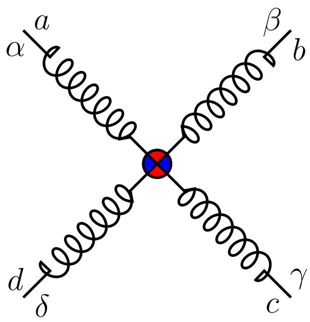
$$= -i\delta_3\delta^{ab}\left(k^2g^{\mu\nu} - k^\mu k^\nu\right). \tag{16}$$

∗ The three-gluon counterterm vertex



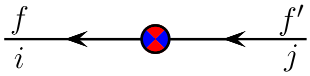
$$= -g\delta_1^{(3g)}\times f^{abc}\left[g^{\alpha\beta}(k_1-k_2)^\gamma + g^{\beta\gamma}(k_2-k_3)^\alpha + g^{\gamma\alpha}(k_3-k_1)^\beta\right]. \tag{17}$$

- The four-gluon counterterm vertex



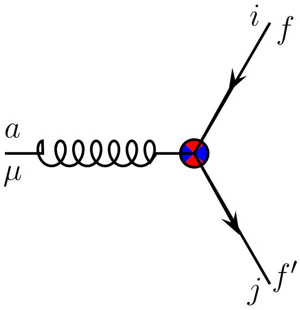
$$= -ig^2\delta_1^{(4g)} \times \begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix}. \tag{18}$$

∗ The two-quark counterterm vertex



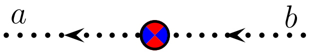
$$= \delta_f^{f'} \delta_i^j \times \left(i\delta_m^{(q_f)} - i\delta_2^{(q_f)} \times \not{p} \right). \tag{19}$$

∗ The quark-gluon counterterm vertex



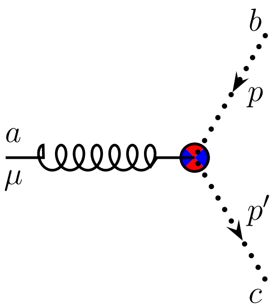
$$= -ig\delta_1^{(q_f)} \delta_f^{f'} \times \gamma^\mu \times (t^a)^j{}_i \,. \tag{20}$$

∗ The two ghost counterterm vertex



$$= \delta^{ab} \times i\delta_2^{(\mathrm{gh})} \times k^2. \tag{21}$$

∗ The ghost-gluon counterterm vertex



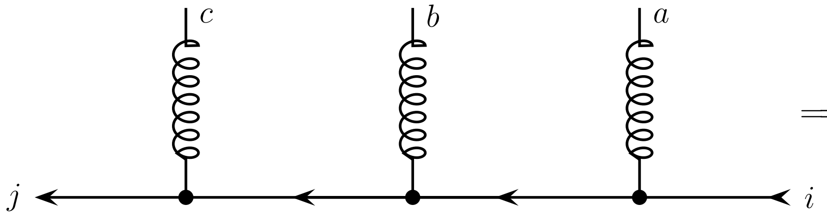
$$= +g\delta_1^{(\mathrm{gh})} \times f^{abc} p'^\mu. \tag{22}$$

In addition to all these propagators and vertices, there are a few general rules:

⋆ Remember that the ghost fields are fermionic, so each closed loop of ghost propagators carries a minus sign.

⋆ The flavor f remains constant along any quark line, open or closed. For an open line, f matches both the incoming and the outgoing quarks (or antiquarks); for closed quark loops, we sum over all the flavors.

⋆ The color of a quark changes from propagator to propagator since the quark-quark-gluon vertices carry the $(t^a)^j{}_i$ factors. In matrix notations, the t^a generators should be multiplied right-to-left in the order of arrows on the quark line, for example



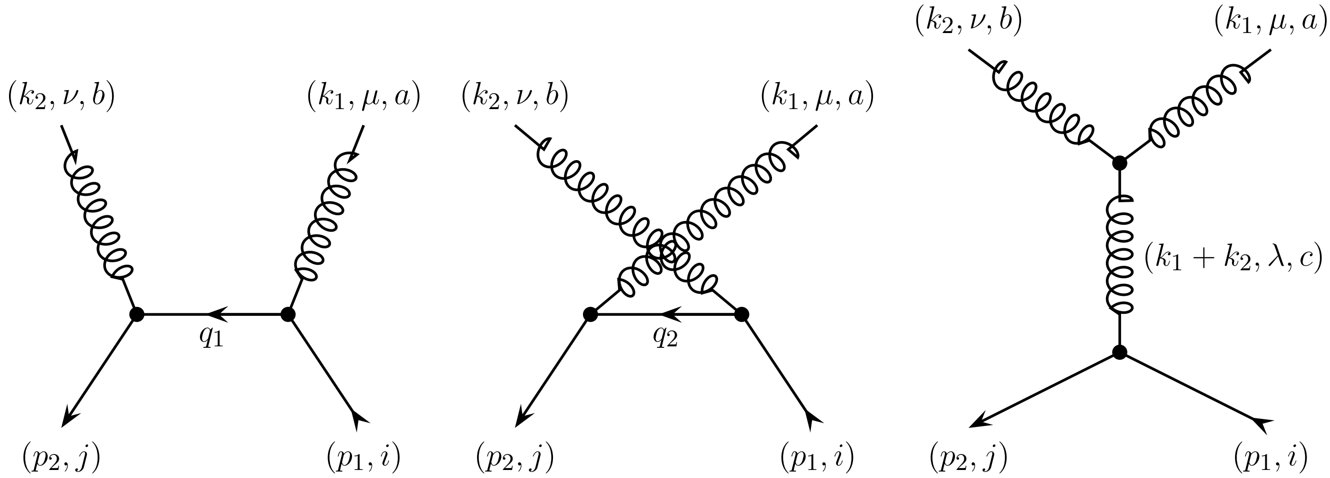
$$\Longrightarrow \quad \left(t^c t^b t^a\right)^j{}_i \times \text{other factors.}$$

For the closed quark lines, one starts at an arbitrary vertex, multiplies all the generators right-to-left in the order of the arrows, than takes the trace over the color indices, $\mathrm{tr}(\cdots t^c t^b t^a)$.

Ward Identities

QCD has weaker Ward identities than QED. In particular, consider the on-shell scattering amplitudes involving the longitudinally polarized gluons. *When one gluon is longitudinal and all other gluons are transverse, the amplitude vanishes. But when two or more gluons are longitudinal, the amplitude does not vanish; instead, it is related to the amplitudes involving the external ghosts instead of the longitudinal gluons.*

As an example, consider the tree level annihilation of a quark and an antiquark into a pair of gluons, $q\bar{q} \to gg$. In QED there are two tree diagrams for the $e^-e^+ \to \gamma\gamma$ annihilation, but in QCD there are three diagrams:



(23)

According to the QCD Feynman rules, these diagrams evaluate to

$$
\begin{aligned}
i\mathcal{M}_1 &= \bar{v}(p_2)\,(-ig\gamma^\nu e^*_{2\nu})\,\frac{i}{\not{q}_1 - m}\,(-ig\gamma^\mu e^*_{1\mu})\,u(p_1) \times \bigl(t^b t^a\bigr)^j_{\;i}\,, && (24)\\
i\mathcal{M}_2 &= \bar{v}(p_2)\,(-ig\gamma^\mu e^*_{1\mu})\,\frac{i}{\not{q}_2 - m}\,(-ig\gamma^\nu e^*_{2\nu})\,u(p_1) \times \bigl(t^a t^b\bigr)^j_{\;i}\,, && (25)\\
i\mathcal{M}_3 &= \bar{v}(p_2)\,(-ig\gamma_\lambda)\,u(p_1) \times (t^c)^j_{\;i} \times \frac{-i}{(k_1+k_2)^2} \times \\
&\quad \times (-g)f^{abc}\bigl[g^{\mu\nu}(-k_1+k_2)^\lambda \;+\; g^{\nu\lambda}(-k_2-(k_1+k_2))^\mu \;+\; g^{\lambda\mu}((k_1+k_2)+k_1)^\nu\bigr] \\
&\qquad\qquad \langle\!\langle\,\text{the 3 gluon vertex; the unusual signs are due to directions of momenta}\,\rangle\!\rangle \\
&\qquad\qquad \langle\!\langle\,\text{the } k_1 \text{ and } k_2 \text{ are outgoing while the } k_3 = k_1+k_2 \text{ is incoming}\,\rangle\!\rangle \\
&\quad \times e^*_{1\mu} e^*_{2\nu}\,, && (26)\\
\mathcal{M}^{\rm net}_{\rm tree} &= \mathcal{M}_1 \;+\; \mathcal{M}_2 \;+\; \mathcal{M}_3\,. && (27)
\end{aligned}
$$

Clearly, each term in the net amplitude is $O(g^2)$ and each term includes the polarization vectors for the two gluons, thus

$$
\mathcal{M} \;=\; e^*_{1\mu} e^*_{2\nu} \times \mathcal{M}^{\mu\nu}\,. \tag{28}
$$

So let us check the Ward identity $k_{1\mu} \times M^{\mu\nu} \stackrel{??}{=} 0$.

For the first diagram's amplitude we have

$$
k_{1\mu} \times \mathcal{M}^{\mu\nu}_1 \;=\; -g^2\bigl(t^b t^a\bigr)^j_{\;i} \times \bar{v}\gamma^\nu\,\frac{1}{\not{q}_1 - m}\,\not{k}_1 u. \tag{29}
$$

In the second factor here, $q_1 = p_1 - k_1$, hence

$$
\frac{1}{\not{q}_1 - m}\,\not{k}_1 \;=\; \frac{1}{\not{q}_1 - m}\,(\not{p}_1 - \not{q}_1) \;=\; -1 \;+\; \frac{1}{\not{q}_1 - m}\,(\not{p}_1 - m), \tag{30}
$$

which for the on-shell quark gives

$$
\frac{1}{\not{q}_1 - m}\,\not{k}_1\,u(p_1) \;=\; -u(p_1) \;+\; \frac{1}{\not{q}_1 - m}\,(\not{p}_1 - m)u(p_1) \;=\; -u(p_1) \;+\; 0 \tag{31}
$$

because $(\not{p}_1 - m)u(p_1) = 0$. Thus,

$$
k_{1\mu} \times \mathcal{M}^{\mu\nu}_1 \;=\; +g^2\bigl(t^b t^a\bigr)^j_{\;i} \times \bar{v}(p_2)\gamma^\nu u(p_1). \tag{32}
$$

Likewise, for the second diagram

$$k_{1\mu} \times \mathcal{M}_2^{\mu\nu} \;=\; -g^2\bigl(t^b t^a\bigr)^j_{\;i} \times \bar{v}\,\not{k}_1 \frac{1}{\not{q}_2 - m}\,\gamma^\nu u. \tag{33}$$

where in the second factor

$$q_2 = k_1 - p_2 \quad\Longrightarrow\quad \not{k}_1 \frac{1}{\not{q}_2 - m} = 1 - (\not{p}_2 + m)\frac{1}{\not{q}_2 - m} \quad\Longrightarrow\quad v(p_2)\,\not{k}_1 \frac{1}{\not{q}_2 - m} = +v(p_2) - 0, \tag{34}$$

thus

$$k_{1\mu} \times \mathcal{M}_2^{\mu\nu} \;=\; -g^2\bigl(t^a t^b\bigr)^j_{\;i} \times \bar{v}(p_2)\gamma^\nu u(p_1). \tag{35}$$

In QED, $k_{1\mu} \times \mathcal{M}_1^{\mu\nu}$ and $k_{1\mu} \times \mathcal{M}_2^{\mu\nu}$ would have canceled each other, but in QCD eqs. (32) and (35) carry different color-dependent factors. So instead of cancellation, we have

$$k_{1\mu} \times \mathcal{M}_{1+2}^{\mu\nu} \;=\; g^2 \bar{v}\gamma^\nu u \times \bigl(t^b t^a - t^a t^b\bigr)^j_{\;i} \;=\; g^2 \bar{v}\gamma^\nu u \times -if^{abc}\bigl(t^c\bigr)^j_{\;i}. \tag{36}$$

But the net color-dependent factor is similar to the third amplitude, so there is a hope that the Ward identity might work when all three diagrams are put together.

For the third diagram we have

$$\begin{aligned} k_{1\mu} \times \mathcal{M}_3^{\mu\nu} \;&=\; -ig^2 f^{abc}\bigl(t^c\bigr)^j_{\;i} \times \bar{v}\gamma^\lambda u \times \frac{1}{(k_1+k_2)^2} \times \\ &\quad \times k_{1\mu} \times \bigl[g^{\mu\nu}(k_2-k_1)^\lambda + g^{\nu\lambda}(-k_1-2k_2)^\mu + g^{\lambda\mu}(2k_1+k_2)^\nu\bigr], \end{aligned} \tag{37}$$

where on the second line

$$\begin{aligned} k_{1\mu} \times [\cdots] \;&=\; k_1^\nu (k_2-k_1)^\lambda + g^{\nu\lambda}(-k_1^2 - 2k_1k_2) + k_1^\lambda (2k_1+k_2)^\nu \\ &=\; g^{\lambda\nu}\bigl(-(k_1+k_2)^2 + k_2^2\bigr) + \bigl[(2-1)k_1^\lambda k_1^\nu + k_1^\lambda k_2^\nu + k_2^\lambda k_1^\nu\bigr] \\ \langle\!\langle\, \text{on shell}\,\rangle\!\rangle & \\ &=\; -g^{\lambda\nu}(k_1+k_2)^2 + (k_1+k_2)^\lambda (k_1+k_2)^\nu - k_2^\lambda k_2^\nu. \end{aligned} \tag{38}$$

Plugging the three terms here back into eq. (37), we obtain

$$k_{1\mu} \times \mathcal{M}_3^{\mu\nu} \;=\; k_{1\mu} \times \mathcal{M}_{3,a}^{\mu\nu} + k_{1\mu} \times \mathcal{M}_{3.b}^{\mu\nu} + k_{1\mu} \times \mathcal{M}_{3,c}^{\mu\nu} \tag{39}$$

where

$$k_{1\mu} \times \mathcal{M}_{3,a}^{\mu\nu} \;=\; +ig^2 f^{abc}\bigl(t^c\bigr)^j_{\;i} \times \bar{v}(p_2)\gamma^\nu u(p_1), \tag{40}$$

$$k_{1\mu} \times \mathcal{M}^{\mu\nu}_{3,b} \;=\; -ig^2 f^{abc} \big(t^c\big)^j_{\;i} \times \bar{v}(p_2)(\not{k}_1 + \not{k}_2) u(p_1) \times \frac{(k_1+k_2)^\nu}{(k_1+k_2)^2}\,, \tag{41}$$

$$k_{1\mu} \times \mathcal{M}^{\mu\nu}_{3,c} \;=\; +ig^2 f^{abc} \big(t^c\big)^j_{\;i} \times \bar{v}(p_2)(\not{k}_2) u(p_1) \times \frac{k_2^\nu}{(k_1+k_2)^2}\,. \tag{42}$$

By inspection of eqs. (40) and (36), the first term in eq. (39) precisely cancels the contributions of the first two diagrams,

$$k_{1\mu}\mathcal{M}^{\mu}_{1+2} \;+\; k_{1\mu} \times \mathcal{M}^{\mu\nu}_{3,a} \;=\; 0. \tag{43}$$

The second term's contribution (41) vanishes for the on-shell quarks. Indeed, by momentum conservation $k_1 + k_2 = p_1 + p_2$, hence

$$\bar{v}(p_2)(\not{k}_1 + \not{k}_2) u(p_1) \;=\; \bar{v}(p_2)(\not{p}_1 + \not{p}_2) u(p_2) \;=\; \bar{v}(p_2)(\not{p}_2 + m) u(p_1) \;+\; \bar{v}(p_2)(\not{p}_1 - m) u(p_1) \;=\; 0 \;+\; 0 \tag{44}$$

and therefore $k_{1\mu} \times \mathcal{M}^{\mu\nu}_{3,b} = 0$.

But the third term's contribution (41) does not vanish, and this breaks the Ward identity for the net QCD amplitude:

$$k_{1\mu} \times \mathcal{M}^{\mu\nu}_{\rm net} \;=\; k_{1\mu} \times \mathcal{M}^{\mu\nu}_{3,c} \;=\; +ig^2 f^{abc} \big(t^c\big)^j_{\;i} \times v\not{k}_2 u \times \frac{1}{(k_1+k_2)^2} \times k_2^\nu \;\neq\; 0. \tag{45}$$

However, the net violation of the Ward identity is proportional to the k_2^ν factor. Therefore, when we contract the amplitude $\mathcal{M}^{\mu\nu}_{\rm net}$ with the polarization vector of the second gluon, we obtain

$$k_{1\mu} \times \mathcal{M}^{\mu\nu}_{\rm net} e^*_{2\nu} \;=\; [\cdots] \times (k_2 e^*_2), \tag{46}$$

which vanishes when the second gluon is transversely polarized! This agrees with the weakened Ward identity of QCD: *Amplitudes involving one longitudinal gluon vanish if all the other gluons are transverse, but if two (or more) gluons are longitudinal, the amplitude does not have to vanish. Instead, such amplitudes are related to the amplitudes involving ghosts and antighosts.*

Indeed, consider the annihilation amplitude of two quarks into two longitudinal gluons, $\mathcal{M}(q\bar{q} \to g_L g_L)$. In Minkowski space, there are two distinct longitudinal polarizations for a gluon moving in the direction $\mathbf{n}$, namely $e^\mu_\pm = (1,\pm\mathbf{n})/\sqrt{2}$. In light of eq. (46), the amplitude for producing two gluons with polarizations $L+$ (*i.e.*, $e^\mu \propto k^\mu$ for each gluon) vanishes, $\mathcal{M}(q\bar q\to g_{L+}g_{L+})=0$. The amplitude for producing two gluons with longitudinal polarizations $L-$ also vanishes, $\mathcal{M}(q\bar q\to g_{L-}g_{L-})=0$, although I am not going to prove it in these notes. Instead, let me focus on the non-zero amplitude for producing one gluon with the longitudinal $L+$ polarization and the other gluons with the longitudinal $L-$ polarization.

In light of eq. (46), we get

$$\mathcal{M}(q\bar q\to g_{L+}g_{Li}) \;=\; \left[e_{1\mu}(L+)\;=\;\frac{k_{1\mu}}{\omega_1\sqrt{2}}\right]\times\mathcal{M}^{\mu\nu}_{\rm net}\times e_{2\nu}(L-)\;=\;\frac{1}{\omega_1\sqrt{2}}\times[\cdots]\times\bigl(e_{2\nu}(L-)k_2^\nu\bigr), \tag{47}$$

where $[\cdots]$ stands for the factors from eq. (45) which I did not write down explicitly in eq. (46), namely

$$[\cdots] \;=\; +ig^2f^{abc}\bigl(t^c\bigr)^j_{\;i}\times v\not{k}_2u\times\frac{1}{(k_1+k_2)^2}\,, \tag{48}$$

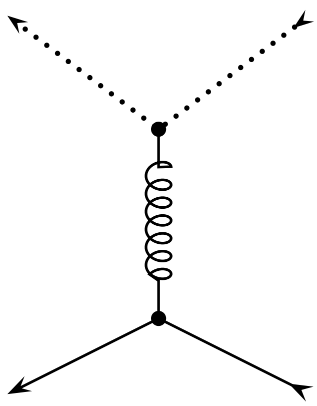
while

$$e_2^\mu(L-)g_{\mu\nu}k_2^\nu \;=\; \frac{(1,-\mathbf{n})^\mu}{\sqrt{2}}\times g_{\mu\nu}\times\omega_2(1,+\mathbf{n})^\nu \;=\; \frac{\omega_2}{\sqrt{2}}\times 2. \tag{49}$$

Thus, in the center of mass frame where $\omega_1=\omega_2=\frac12 E_{\rm cm}$ while $(k_1+k_2)^2=s=E^2_{\rm cm}$, we have

$$\mathcal{M}(q+\bar q\to g_{L+}+g_{L-}) \;=\; \frac{ig^2}{s}\,f^{abc}\bigl(t^c\bigr)^j_{\;i}\times v(p_2)\not{k}_2u(p_1). \tag{50}$$

Let's compare this amplitude to the annihilation of the same quark and the same antiquark into a ghost and antighost. At the tree level, there is only one diagram for the later process,



which yields the amplitude

$$\begin{aligned} i\mathcal{M}_{\rm tree}(q+\bar q\to {\rm gh}+\overline{\rm gh}) &= \bar v(p_2)(-ig\gamma^\lambda)u(p_1)\times\left(t^c\right)^j_{\ i}\times\frac{-ig_{\lambda\nu}}{s}\times gf^{abc}k_2^\nu \\ &= -\frac{g^2}{s}\,f^{abc}\left(t^c\right)^j_{\ i}\times v(p_2)\not{k}_2u(p_1). \end{aligned} \tag{51}$$

By inspection, this amplitude is equal to the amplitude (50) for $q+\bar q$ annihilating into two longitudinal gluons instead of a ghost and an antighost,

$$\mathcal{M}(q+\bar q\to {\rm gh}+\overline{\rm gh}) \;=\; \mathcal{M}(q+\bar q\to g_{L+}+g_{L-}). \tag{52}$$

In the next set of notes we shall learn that such relations stem from the BRST symmetry, but right now we may use eq. (52) to understand how the physical cross-sections work in QCD.

The ghosts violate the spin-statistics theorem, so we must give up one one of its assumptions: relativity, positive particle energies, or the positive norm in the Hilbert space. The correct choice is to give up on the norm positivity in the extended Hilbert space including both the physical and the unphysical quanta: While the physical (anti)quarks and transverse gluons must have positive norm, the norm for the unphysical longitudinal gluons is ghost has mixed signature — positive for the longitudinal gluons but negative for the ghosts and antighosts. And because of the negative norm for the (anti)ghosts states, the cross-section for the annihilation-into-ghosts process comes out negative,

$$\frac{d\sigma}{d\Omega} \;=\; -\frac{|\mathcal{M}|^2}{64\pi^2 s}\,. \tag{53}$$

By themselves, the negative cross-sections are impossible, but they make sense in the context of net unpolarized cross-section where the final states could be either gluons or ghosts,

$$\frac{d\sigma(q+\bar q\to\cdots)}{d\Omega} \;=\; \frac{d\sigma(q+\bar q\to g_T+g_T)}{d\Omega}+\frac{d\sigma(q+\bar q\to g_L+g_L)}{d\Omega}+\frac{d\sigma(q+\bar q\to {\rm gh}+\overline{\rm gh})}{d\Omega}\,. \tag{54}$$

Thanks to eq. (52), the negative cross-section for the annihilation into ghosts precisely cancels the positive cross-section for the annihilation into longitudinal gluons,

$$\frac{d\sigma(q+\bar q\to g_L+g_L)}{d\Omega}+\frac{d\sigma(q+\bar q\to {\rm gh}+\overline{\rm gh})}{d\Omega} \;=\; 0. \tag{55}$$

Thus, the un-physical processes cancel each other, and the net annihilation cross-section is just the cross-section for producing the physical states only. At the $O(g^4)$ level, this means annihilation into a pair of transverse gluons only,

$$\frac{d\sigma(q+\bar{q}\rightarrow g+g\,\mathrm{or}\,\mathrm{gh}+\overline{\mathrm{gh}})}{d\Omega} = \frac{d\sigma(q+\bar{q}\rightarrow g_T+g_T\,\mathrm{only})}{d\Omega}\,. \tag{56}$$

Note: this relation is important for the unitarity of QCD.