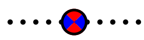
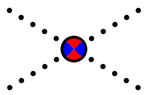
Problem **1** — the textbook problem **10.2**(b):

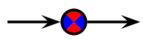
The previous homework set #16 included part (a) of the textbook problem **10.2**. In the solutions to that part, I showed that the Yukawa theory needs 6 counterterms δ_m^ϕ, δ_Z^ϕ, δ_m^ψ, δ_Z^ψ, δ_g, and δ_λ, which lead to four counterterm vertices:



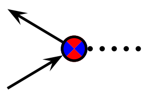
$$= -i\delta_m^\phi \;+\; ip^2\,\delta_Z^\phi\,,$$



$$= -i\delta_\lambda\,,$$



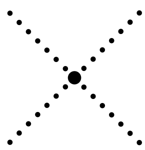
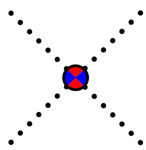
$$= -i\delta_m^\psi \;+\; i\,\not{p}\,\delta_Z^\psi\,,$$

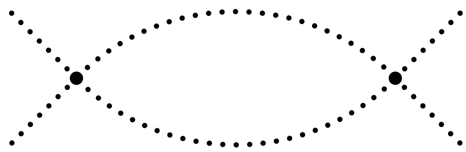


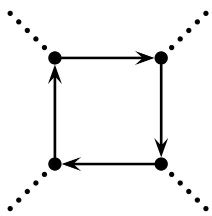
$$= -\delta_g\gamma^5 \qquad \text{(S.1)}$$

(For details, see the solutions to the previous homework, set#16.) In this part (b) we shall calculate the infinite parts of all the counterterms.

Let's start with the δ_λ counterterm which cancels the divergence of the four-scalar 1PI amplitude $\mathcal{V}(k_1, k_2, k_3, k_4)$. At the one-loop level of analysis, we have the following Feynman diagrams:

$i\mathcal{M}^{1\,\mathrm{loop}}(k_1, k_2, k_3, k_4) \;=$  $+$ 

$+$  $+$ two similar

$+$  $+$ five similar. (S.2)

The *similar* diagrams here are related by non-trivial permutations of the external legs. For the scalar loops, non-trivial means different pairing of the external legs at the vertices (modulo vertex permutations), hence 3 distinct diagrams, while for the fermionic loops non-trivial means different *cyclic order* of the 4 legs, hence 6 distinct diagrams.

Since we have done the scalar loops in class, let's focus on the fermionic loop at the bottom line of (S.2). Each such loop yields

$$-\int\frac{d^4p_1}{(2\pi)^4}\,\mathrm{Tr}\left\{(-g\gamma^5)\,\frac{i}{\not p_1-M+i0}\,(-g\gamma^5)\,\frac{i}{\not p_2-M+i0}\,(-g\gamma^5)\,\frac{i}{\not p_3-M+i0}\,(-g\gamma^5)\,\frac{i}{\not p_4-M+i0}\right\} \tag{S.3}$$

where

$$p_2=p_1+k_1\,,\qquad p_3=p_2+k_2,\qquad p_4=p_3+k_3,\quad\text{and}\quad p_1=p_4+k_4\,.$$

For generic values of the external momenta $k_1,\ldots,k_4$, the integral (S.3) is quite complicated, but its divergence is k-independent and hence may be evaluated for any particular choice of k_i we find convenient. Clearly, the simplest set of the k_i is $k_1=k_2=k_3=k_4=0$; this is off-shell, but that's OK. Consequently, the integral (S.3) becomes

$$\begin{aligned} i\mathcal{V}^{\psi\,\mathrm{loop}}(0,0,0,0) &= -\int\frac{d^4p_1}{(2\pi)^4}\,\mathrm{tr}\left[\left((-g\gamma^5)\,\frac{i}{\not p-M+i0}\right)^4\right] \\ &= -g^4\int\frac{d^4p_1}{(2\pi)^4}\,\frac{\mathrm{tr}\left[\left(\gamma^5(\not p+M)\right)^4\right]}{(p^2-M^2+i0)^4} \\ &= -g^4\int\frac{d^4p_1}{(2\pi)^4}\,\frac{4}{(p^2-M^2+i0)^2} \end{aligned} \tag{S.4}$$

where the last equality follows from

$$\left(\gamma^5(\not p+M)\right)^2 \;=\; \gamma^5(\not p+M)\gamma^5(\not p+M) \;=\; (-\not p+M)(\not p+M) \;=\; -p^2\;+\;M^2 \tag{S.5}$$

and hence

$$\mathrm{tr}\left[\left(\gamma^5(\not p+M)\right)^4\right] \;=\; 4(p^2-M^2)^2. \tag{S.6}$$

Evaluating the integral on the last line of eq. (S.4) using dimensional regularization, we obtain

$$\mathcal{V}_{\psi\,\mathrm{loop}}(k_1=k_2=k_3=k_4=0) \;=\; \frac{-4g^4}{16\pi^2}\left(\frac{1}{\bar\epsilon}+\log\frac{\mu^2}{M^2}\right) \tag{S.7}$$

where

$$\frac{1}{\bar{\epsilon}} \stackrel{\text{def}}{=} \frac{1}{\epsilon} - \gamma_E + \log(4\pi). \tag{S.8}$$

This notation is common in dimensional regularization: because the $1/\epsilon$ divergence is usually accompanied by the finite $-\gamma_E + \log(4\pi)$ constant, it's convenient to combine them into a single term denoted $1/\bar{\epsilon}$.

It remains to multiply the amplitude (S.7) by 6 (for six similar diagrams) and add contributions of the other diagrams (S.2). The latter diagrams have been evaluated in class in the context of the scalar $\lambda\Phi^4$ theory, thus to order $O(\lambda^2$ or $g^4)$,

$$\mathcal{V}(k_1 = k_2 = k_3 = k_4 = 0) = -\lambda - \delta_\lambda + \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{m^2}\right) - \frac{24g^4}{16\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{M^2}\right). \tag{S.9}$$

The renormalization condition for the physical λ coupling is the on-shell four-particle amplitude $\mathcal{M}(\text{threshold}) = -\lambda$, or in other words $\mathcal{V} = -\lambda$ when all external momenta are on shell and at the threshold ($s = 4m^2$, $t = u = 0$). At other values of external momenta, we should have

$$\mathcal{V}(k_1, k_2, k_3, k_4) = -\lambda - \frac{\lambda^2}{32\pi^2} \times \text{finite} - \frac{4g^4}{16\pi^2} \times \text{finite} + \text{higher loop orders}. \tag{S.10}$$

Comparing this formula with eq. (S.9) gives us

$$\delta_\lambda^{\text{1 loop}} = \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{m^2} + \text{finite}\right) - \frac{24g^4}{16\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{M^2} + \text{finite}\right). \tag{S.11}$$

As promised last week, the fermionic loops provide for $\delta_\lambda \neq 0$ even if were to start from $\lambda = 0$.

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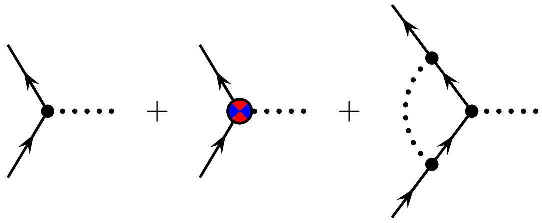
Next, we want to calculate the δ_g counterterm, so let us consider the $\Phi\overline{\Psi}\gamma^5\Psi$ vertex correction. By analogy with the QED vertex, we denote $\Gamma^{(5)}(p', p)$ the 1PI amplitude for two fermions of respective momenta p and p' and one pseudoscalar of momentum $k = p' - p$. At the one-loop level of analysis,

$$-\Gamma^{(5)}(p', p) \;=\;$$  (S.12)

$$= -g\gamma^5 - \delta_g \gamma^5$$

$$+ \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i0} \times (-g\gamma^5) \frac{i}{\not p' + \not q - M + i0} (-g\gamma^5) \frac{i}{\not p + \not q - M + i0} (-g\gamma^5).$$

As in the previous calculation, the loop integral here diverges logarithmically, and the divergent part does not depend on the external momenta. Consequently, we may calculate this divergence for any values of p, p', and $k = p' - p$ we like, for example $p = p' = k = 0$, which makes for a much simpler integral. Indeed, for zero external momenta, the fermionic line becomes

$$\begin{aligned}(-g\gamma^5) \frac{i}{0 + \not q - M + i0} (-g\gamma^5) \frac{i}{0 + \not q - M + i0} (-g\gamma^5) &= g^3 \frac{\gamma^5(\not q + M)\gamma^5(\not q + M)\gamma^5}{(q^2 - M^2 + i0)^2} \\ &= g^3 \frac{-\gamma^5}{q^2 - M^2 + i0}\end{aligned} \tag{S.13}$$

where the second equality follows from eq. (S.5). Consequently, the loop integral in eq. (S.12) becomes easy to evaluate:

$$\begin{aligned}\int \frac{d^4 q}{(2\pi)^4} \frac{-ig^3\gamma^5}{(q^2 - m^2 + i0)(q^2 - M^2 + i0)} &= g^3\gamma^5 \times \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{(q_E^2 + m^2)(q_E^2 + M^2)} \\ &= g^3\gamma^5 \times \int_0^1 dx \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{[q_e^2 + xM^2 + (1-x)m^2]^2}\end{aligned}$$

⟨⟨using dimensional regularization⟩⟩

$$\begin{aligned}&= \frac{g^3\gamma^5}{16\pi^2} \int_0^1 dx \left(\frac{1}{\bar\epsilon} + \log\frac{\mu^2}{xM^2 + (1-x)m^2}\right) \\ &= \frac{g^3\gamma^5}{16\pi^2} \left(\frac{1}{\bar\epsilon} + \log\frac{\mu^2}{M^2} + 1 - \frac{m^2}{M^2 - m^2} \log\frac{M^2}{m^2}\right).\end{aligned} \tag{S.14}$$

Thus, to the order g^3,

$$\Gamma^{(5)}(p'=p=0) \;=\; -g\gamma^5 \;-\; \delta_g\gamma^5 \;+\; \frac{g^3\gamma^5}{16\pi^2}\left(\frac{1}{\bar\epsilon} \;+\; \log\frac{\mu^2}{M^2} \;+\; \text{finite}\right). \tag{S.15}$$

And since the divergent part is momentum independent, it follows that for any external momenta,

$$\Gamma^{(5)}(p',p) \;=\; -g\gamma^5 \;-\; \delta_g\gamma^5 \;+\; \frac{g^3\gamma^5}{16\pi^2}\left(\frac{1}{\bar\epsilon} \;+\; \log\frac{\mu^2}{M^2} \;+\; \text{finite function of}(p',p)\right) \;+\; O(g^5 \text{ or } g^3\lambda). \tag{S.16}$$

In class, I have not explained the renormalization condition for the Yukawa coupling g, but it's clear that such condition should have form $\Gamma^{(5)} = -g\gamma^5$ for the on-shell fermions and some particular value of the pseudoscalar's q^2, for example $q^2 = 0$ or on-shell $q^2 = m^2$ (allowed for $m \geq 2M$). In light of eq. (S.16), this means

$$\delta_g^{1\,\text{loop}} \;=\; \frac{g^3}{16\pi^2}\left(\frac{1}{\bar\epsilon} \;+\; \log\frac{\mu^2}{M^2} \;+\; \text{finite}\right) \tag{S.17}$$

where the finite part depends on the specific renormalization condition (and in general is a painfully complicated function of the m/M mass ratio), but the infinite part is clear and unambiguous.

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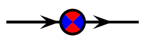
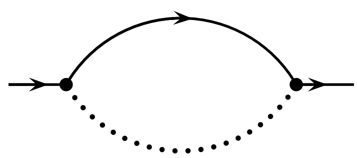
Our next targets are the fermion's mass and kinetic energy counterterms δ^ψ_M and δ^ψ_Z. At the one-loop level of analysis, the Dirac field's 1PI two-point Green's function is

$-i\Sigma_\psi^{1\,\text{loop order}}(\not{p}) \;=$  $+$ 

$$= \; -i\delta^\psi_M \;+\; i\delta^\psi_Z\,\not{p} \;+ \int\!\frac{d^4k}{(2\pi)^4}\,\frac{i}{q^2-m^2+i0}\times(-g\gamma^5)\,\frac{i}{\not{p}+\not{q}-M+i0}\,(-g\gamma^5). \tag{S.18}$$

This time, we cannot set $p=0$ so we must be more careful. Let us re-write the integrand of the loop integral as

$$-g^2\,\frac{\mathcal{N}}{\mathcal{D}}\tag{S.19}$$

where

$$\mathcal{N}\ =\ \gamma^5(\not{q}+\not{p}+M)\gamma^5\quad\text{and}\quad\mathcal{D}\ =\ \Big(q^2-m^2+i0\Big)\times\Big((p+q)^2-M^2+i0\Big).\tag{S.20}$$

Using the Feynman's parameter trick we may simplify the denominator as

$$\begin{aligned}\frac{1}{\mathcal{D}}\ &=\int_0^1\!dx\frac{1}{\big[(1-x)(q^2-m^2)\ +\ x((p+q)^2-M^2)\ +\ i0\big]^2}\\ &=\int_0^1\!dx\frac{1}{\big[\ell^2\ -\ \Delta\ +\ i0\big]^2}\end{aligned}\tag{S.21}$$

where

$$\ell\ =\ q\ +\ xp\quad\text{and}\quad\Delta\ =\ xM^2\ +\ (1-x)m^2\ -\ x(1-x)p^2.\tag{S.22}$$

As usual, we take the $\int dx$ integral after integrating over momentum, which allows us to shift the momentum variable from p^μ to ℓ^μ, thus

$$\Sigma_\psi^{1\,\text{loop}}(\not{p})\ \ =\ -\ ig^2\int_0^1\!dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}}{[\ell^2-\Delta+i0]^2}\,.\tag{S.23}$$

In terms of the shifted loop momentum ℓ, the numerator becomes

$$\mathcal{N}\ =\ \gamma^5\Big(\not{\ell}+\,(1-x)\,\not{p}\,+\,M\Big)\gamma^5\ =\ M\ -\ (1-x)\,\not{p}-\,\not{\ell},\tag{S.24}$$

where the last term $\not{\ell}$ does not contribute to the momentum integral because it's odd under the $\ell\to-\ell$ symmetry, thus

$$\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\not{\ell}}{[\ell^2-\Delta+i0]^2}\ \ =\ 0\tag{S.25}$$

and therefore

$$\Sigma_\psi^{1\,\text{loop}}(\not{p})\ \ =\ -ig^2\int_0^1\!dx\,[M-(1-x)\,\not{p}\,]\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i0)^2}\,.\tag{S.26}$$

Note that although the two-fermion amplitude Σ_ψ has superficial degree of divergence $D=+1$, the leading linear divergence (S.25) vanishes by Lorentz symmetry, and the remaining momentum integral (S.26) has only the sub-leading logarithmic UV divergence. Evaluating this integral by going to the Euclidean momentum space and using dimensional regularization, we obtain

$$\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i0)^2}\;=\;\frac{i}{16\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right),\tag{S.27}$$

and therefore

$$\Sigma^{1\,\text{loop}}_{\psi}(\not p)\;=\;\delta^{\psi}_{M}\,-\,\delta^{\psi}_{Z}\,\not p+\frac{g^2}{16\pi^2}\int\limits_0^1\!dx\,[M-(1-x)\not p]\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{(1-x)m^2+xM^2-x(1-x)p^2}\right).\tag{S.28}$$

The renormalization conditions for the fermion's propagator correction $\Sigma^{\psi}(\not p)$ are

$$\Sigma\Big|_{\not p\,=\,M}\;=\;0\quad\text{and}\quad\frac{d\Sigma}{d\not p}\Big|_{\not p\,=\,M}\;=\;0.\tag{S.29}$$

In light of eq. (S.28), the second condition (S.29) becomes

$$\begin{aligned}\delta^{\psi}_{Z}[1\,\text{loop}]\;&=\;\frac{g^2}{16\pi^2}\,\frac{\partial}{\partial\not p}\int\limits_0^1\!dx\,[M-(1-x)\not p]\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{(1-x)m^2+xM^2-x(1-x)p^2}\right)\Bigg|_{\not p\,=\,M}\\&=\;\frac{g^2}{16\pi^2}\int\limits_0^1\!dx\left[(x-1)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{x^2M^2+(1-x)m^2}\right)\,+\,\frac{2x^2(1-x)M^2}{x^2M^2+(1-x)m^2}\right]\\&=\;-\frac{g^2}{32\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2}\,+\,\text{finite}\right).\end{aligned}\tag{S.30}$$

At the same time, the first condition (S.29) implies

$$\begin{aligned}&\delta^{\psi}_{M}[1\,\text{loop}]\;-\;M\delta^{\psi}_{Z}[1\,\text{loop}]\\&\quad=\;-\frac{g^2}{16\pi^2}\int\limits_0^1\!dx\,[M-(1-x)\not p]\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{(1-x)m^2+xM^2-x(1-x)p^2}\right)\Bigg|_{\not p\,=\,M}\\&\quad=\;-\frac{g^2}{16\pi^2}\int\limits_0^1\!dx\,xM\times\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{x^2M^2+(1-x)m^2}\right)\\&\quad=\;-\frac{g^2M}{32\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2}\,+\,\text{finite}\right)\end{aligned}\tag{S.31}$$

and consequently

$$\delta_M^\psi[1\,\text{loop}] \;=\; -\frac{g^2 M}{16\pi^2}\left(\frac{1}{\bar\epsilon} \;+\; \log\frac{\mu^2}{M^2} \;+\; \text{finite}\right). \tag{S.32}$$

Note that similarly to QED, the fermionic mass counterterm in the Yukawa theory is proportional to the mass itself and diverges logarithmically rather than linearly in the UV cutoff (*cf.* integral (S.26) prior to dimensional regularization). As in QED, this behavior is due to an additional symmetry of the Yukawa theory when the fermion mass happens to vanish. Specifically, for $M = 0$ we have a *discrete chiral symmetry*

$$\Psi(x) \;\to\; \gamma^5\Psi(x), \quad \overline{\Psi}(x) \;\to\; -\overline{\Psi}(x)\gamma^5, \quad \Phi(x) \;\to\; -\Phi(x). \tag{S.33}$$

Unlike the gauge coupling in QED, the pseudoscalar Yukawa coupling does not respect continuous chiral transforms $\Psi(x) \to \exp(i\alpha\gamma^5)\Psi(x)$, but the discrete symmetry is sufficient for preventing the massless Yukawa theory from developing a mass shift via loop corrections.

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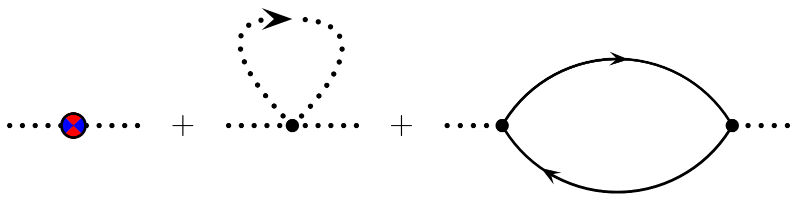
Finally, consider the boson's mass and kinetic energy counterterms δ_M^ϕ and δ_Z^ϕ. At the one-loop level of analysis, the pseudoscalar field's 1PI two-point Green's function is

$$-i\Sigma_\phi^{1\,\text{loop}}(k^2) \;=$$



$$= \; -i\delta_m^\phi \;+\; i\delta_Z^\phi\, k^2 \;+\; \frac{i\lambda m^2}{32\pi^2}\left(\frac{1}{\bar\epsilon} + 1 + \log\frac{\mu^2}{m^2}\right) - \int\!\frac{d^4p}{(2\pi)^4}\,\text{tr}\left(\frac{i}{\not{p} - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{p} + \not{k} - M + i0}\,(-g\gamma^5)\right). \tag{S.34}$$

Again, we re-write the fermionic loop integral as

$$+\, g^2\int\!\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}} \tag{S.35}$$

where the denominator is the usual

$$\mathcal{D} \;=\; \Big(p^2 - M^2 + i0\Big) \times \Big((p+k)^2 - M^2 + i0\Big) \tag{S.36}$$

and hence

$$\begin{aligned} \frac{1}{\mathcal{D}} &= \int_0^1 dx \, \frac{1}{[\ell^2 - \Delta + i0]^2} \\ \text{for } \ell &= p \;+\; kx \\ \text{and } \Delta &= M^2 \;-\; x(1-x)k^2, \end{aligned} \tag{S.37}$$

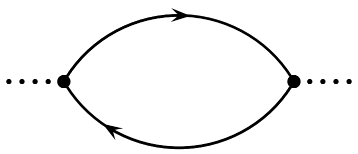
and the numerator is

$$\begin{aligned} \mathcal{N} &= \operatorname{tr}\big[(\not p + M)\gamma^5(\not p + \not k + M)\gamma^5\big] \\ &= \operatorname{tr}\big[(M + \not p)(M - \not p - \not k)\big] \\ &= 4M^2 \;-\; 4p(p+k) \\ &= 4M^2 \;-\; 4(\ell - xk)(\ell + k - xk) \\ &= 4M^2 \;-\; 4\ell^2 \;+\; 4x(1-x)k^2 \;-4(1-2x)(\ell\cdot k). \end{aligned} \tag{S.38}$$

The last term here is odd with respect to $\ell \to -\ell$ and hence does not contribute to the $\int d^4\ell$ integral. Effectively

$$\mathcal{N} \;\cong\; 4M^2 \;-\; 4\ell^2 \;+\; 4x(1-x)k^2\,,$$

so the integral (S.35) becomes



$$\begin{aligned} &= 4g^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \, \frac{M^2 + x(1-x)k^2 - \ell^2}{(\ell^2 - \Delta + i0)^2} \\ &= 4ig^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4} \, \frac{2M^2 - \Delta + \ell_E^2}{(\ell_E^2 + \Delta)^2}\,. \end{aligned} \tag{S.39}$$

In four dimensions, the momentum integral (S.39) diverges quadratically. Hence, in dimensional regularization, we need to analytically continue from $D = 4$ Euclidean dimensions down to

$D < 2$, evaluate the integral for $D < 2$, and only then continue back to $D = 4-2\epsilon$. Thus, working in the Euclidean momentum space, we have

$$\begin{aligned}
\int\frac{d^4\ell}{(2\pi)^4}\,\frac{2M^2-\Delta+\ell^2}{(\ell^2+\Delta)^2} \;&\longrightarrow\; \mu^{4-D}\int\frac{d^D\ell}{(2\pi)^D}\,\frac{2M^2-\Delta+\ell^2}{(\ell^2+\Delta)^2}\\
&= \mu^{4-D}\int\frac{d^D\ell}{(2\pi)^D}\int\limits_0^\infty dt\,t\,e^{-t\Delta}\left(2M^2-\Delta-\frac{\partial}{\partial t}\right)e^{-t\ell^2}\\
&= \mu^{4-D}\int\limits_0^\infty dt\,t\,e^{-t\Delta}\left(2M^2-\Delta-\frac{\partial}{\partial t}\right)\left[\int\frac{d^D\ell_E}{(2\pi)^D}\,e^{-t\ell_E^2} \;=\; (4\pi t)^{-D/2}\right]\\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\int\limits_0^\infty dt\,t\,e^{-t\Delta}\left((2M^2-\Delta)t^{-(D/2)} \;+\; \tfrac{D}{2}\,t^{-(D/2)-1}\right)\\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\left((2M^2-\Delta)\Gamma(2-\tfrac{D}{2})\Delta^{(D/2)-2} \;+\; \tfrac{D}{2}\Gamma(1-\tfrac{D}{2})\Delta^{(D/2)-1}\right)\\
\langle\!\langle\text{now take } D=4-2\epsilon\rangle\!\rangle\qquad&\\
&= \frac{1}{16\pi^2}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}\left(2M^2\;-\;\Delta\;+\;\frac{2-\epsilon}{\epsilon-1}\Delta\right)\\
&\underset{\epsilon\to 0}{\longrightarrow}\;\frac{1}{16\pi^2}\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{\Delta}\right)\;-\;\Delta\right].
\end{aligned}\tag{S.40}$$

Consequently,

$$\begin{aligned}
\Sigma_\phi^{1\,\text{loop}}(k^2) \;&=\; \delta_m^\phi \;-\; \delta_Z^\phi\,k^2 \;-\; \frac{\lambda m^2}{32\pi^2}\left(\frac{1}{\bar\epsilon}\;+\;1\;+\;\log\frac{\mu^2}{m^2}\right)\\
&\quad-\;\frac{g^2}{4\pi^2}\int\limits_0^1 dx\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{\Delta}\right)\;-\;\Delta\right].
\end{aligned}\tag{S.41}$$

Similarly to the fermion's propagator correction Σ_ψ discussed above, the renormalization conditions for a scalar or a pseudoscalar field are

$$\Sigma_\phi\Big|_{k^2=m^2} \;=\; 0 \qquad\text{and}\qquad \frac{\partial\Sigma_\phi}{\partial k^2}\Big|_{k^2=m^2} \;=\; 0. \tag{S.42}$$

Therefore, in light of eq. (S.41),

$$\begin{aligned}
\delta_Z^\phi[1\,\text{loop}] &= -\frac{g^2}{4\pi^2}\frac{\partial}{\partial k^2}\int\limits_0^1 dx\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\,-\,\Delta\right]_{k^2=m^2}\\
&= +\frac{g^2}{4\pi^2}\int\limits_0^1 dx\,x(1-x)\times\frac{\partial}{\partial\Delta}\left((2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\,-\,\Delta\right)\bigg|_{k^2=m^2}\\
&= -\frac{g^2}{4\pi^2}\int\limits_0^1 dx\,x(1-x)\left[\frac{3}{\bar\epsilon}\,+\,3\log\frac{\mu^2}{M^2-x(1-x)m^2}\,+\,\frac{2x(1-x)k^2}{M^2-x(1-x)m^2}\right]\\
&= -\frac{g^2}{8\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2}\,+\,\text{finite}\right).
\end{aligned}\tag{S.43}$$

Likewise,

$$\begin{aligned}
&\delta_m^\phi[1\,\text{loop}]\,-\,m^2\,\delta_Z^\phi[1\,\text{loop}]\,-\,\frac{\lambda m^2}{32\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,1\,+\,\log\frac{\mu^2}{m^2}\right)\\
&\quad= -\frac{g^2}{4\pi^2}\int\limits_0^1 dx\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\,-\,\Delta\right]_{k^2=m^2}\\
&\quad= -\frac{g^2}{4\pi^2}\int\limits_0^1 dx\left[\Big(3x(1-x)m^2-M^2\Big)\times\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2-x(1-x)m^2}\right)+\,\text{finite}\right]\\
&\quad= -\frac{g^2}{4\pi^2}\left((\tfrac{1}{2}m^2-M^2)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2}\right)+\,\text{finite}\right]
\end{aligned}\tag{S.44}$$

and hence

$$\delta_m^\phi[1\,\text{loop}]\;=\;\left[\frac{\lambda m^2}{32\pi^2}\,+\,\frac{g^2M^2}{4\pi^2}\right]\times\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2}\right)\,+\,\text{finite}.\tag{S.45}$$

Note that unlike the other counterterms of the Yukawa theory, the pseudoscalar mass correction δ_m^ϕ diverges quadratically rather than logarithmically. The dimensional regularization however does not see the quadratic divergence itself, all it sees is the sub-leading logarithmic divergence accompanying the quadratic divergence. Thus, in terms of a different UV cutoff, eq. (S.45) means

$$\delta_m^\phi[1\,\text{loop}]\;=\;(\text{unknown})\times\Lambda^2\,+\,\left[\frac{\lambda m^2}{32\pi^2}\,+\,\frac{g^2M^2}{4\pi^2}\right]\times\log\frac{\Lambda^2}{M^2}\,+\,\text{finite},\tag{S.46}$$

where the coefficient of the leading Λ^2 divergence depends on the cutoff's details — such as the exact definition of Λ^2 for each cutoff. FYI, for the Wilson's hard-edge cutoff

$$\delta_m^\phi[1\,\text{loop}] \;=\; -\frac{\lambda}{32\pi^2}\left(\Lambda^2 \,-\, m^2\log\frac{\Lambda^2}{m^2}\right) \;-\; \frac{g^2}{4\pi^2}\left(\Lambda^2 \,-\, M^2\log\frac{\Lambda^2}{M^2}\right) \;+\text{finite}. \tag{S.47}$$