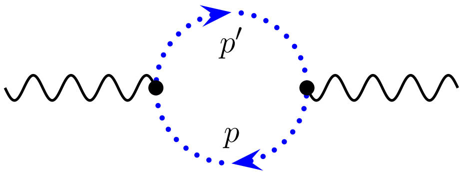
Problem **2**(a):

Let us start with the first diagram (1):



(S.1)

Direct evaluation of the Feynman rules gives us

$$i\Sigma^{\mu\nu}_{(1)}(k) \;=\; \int\frac{d^4p}{(2\pi)^4}\, ie(p'+p)^\mu \times \frac{i}{p^2-M^2+i0} \times ie(p+p')^\nu \times \frac{i}{p'^2-M^2+i0} \tag{S.2}$$

where $p'=p+k$. As usual, we combine the two denominators using Feynman parameter trick, thus

$$\frac{1}{p^2-M^2+i0}\times\frac{1}{(p+k)^2-M^2+i0} \;=\; \int\limits_0^1 dx\,\frac{1}{[q^2-\Delta+i0]^2} \tag{S.3}$$

where

$$q^2\,-\,\Delta \;=\; x(p+k)^2\,+\,(1-x)p^2\,-\,M^2 \tag{S.4}$$

and hence

$$q \;=\; p\,+\,xk \quad\text{and}\quad \Delta \;=\; M^2\,-\,x(1-x)k^2. \tag{S.5}$$

Next, we shift the integration variable from p to q, and this gives us

$$\Sigma^{\mu\nu}_{(1)}(k) \;=\; -ie^2\int\limits_0^1 dx\int\frac{d^4q}{(2\pi)^4}\,\frac{(p+p')^\mu(p+p')^\nu}{[q^2-\Delta+i0]^2} \tag{S.6}$$

where in the numerator

$$\begin{aligned}(p+p')^\mu(p+p')^\nu \;&=\; (2q+(1-2x)k)^\mu(2q+(1-2x)k)^\nu\\ &=\; 4q^\mu q^\nu \;+\; (1-2x)^2k^\mu k^\nu \;+2(1-2x)[q^\mu k^\nu+k^\mu q^\nu]\end{aligned} \tag{S.7}$$

Note that the last term on the second line here is odd with respect to q and hence does not contribute to the $\int dq$ integral. As to the first term on the second line, in the context of $\int dq$ integral $q^\mu q^\nu$ is equivalent to $g^{\mu\nu} \times q^2/D$. Altogether, we have

$$\begin{aligned}(p+p')^\mu(p+p')^\nu \;&\cong\; \frac{4}{D}\,q^2 \times g^{\mu\nu} \;+\; (1-2x)^2 k^\mu k^\nu \\ &=\; \bigl(k^\mu k^\nu \;-\; k^2 g^{\mu\nu}\bigr) \times (1-2x)^2 +\; g^{\mu\nu} \times \left(\frac{4}{D}\,q^2 \;+\; (1-2x)^2 k^2\right),\end{aligned} \tag{S.8}$$

and consequently

$$\Sigma^{\mu\nu}_{(1)}(k) \;=\; \bigl(k^2 g^{\mu\nu} \;-\; k^\mu k^\nu\bigr) \times \Pi_{(1)}(k^2) \;+\; g^{\mu\nu} \times \Xi_{(1)}(k^2) \tag{S.9}$$

where

$$\Pi_{(1)}(k^2) \;=\; ie^2 \int_0^1 dx\,(1-2x)^2 \int_{\text{reg}} \frac{d^4q}{(2\pi)^4}\, \frac{1}{\bigl(q^2-\Delta+i0\bigr)^2} \tag{S.10}$$

and

$$\Xi_{(1)}(k^2) \;=\; -ie^2 \int_0^1 dx \int_{\text{reg}} \frac{d^4q}{(2\pi)^4}\, \frac{\frac{4}{D}\,q^2 \;+\; (1-2x)^2 k^2}{\bigl(q^2-\Delta+i0\bigr)^2}\,. \tag{S.11}$$

Our first task is to verify the tensor structure of the two-photon amplitude, so let us focus on the coefficient $\Xi_{(1)}$ of the wrong tensor. Applying Wick rotation and dimensional regularization to the momentum integral in eq. (S.11), we calculate

$$\begin{aligned} -i \int_{\text{reg}} \frac{d^4q}{(2\pi)^4}\, \frac{\frac{4}{D}q^2 \;+\; (1-2x)^2 k^2}{\bigl(q^2-\Delta+i0\bigr)^2} \;&= \\ &= \int_{\text{reg}} \frac{d^4q_E}{(2\pi)^4}\, \frac{-\frac{4}{D}q_E^2 \;+\; (1-2x)^2 k^2}{\bigl(q_E^2+\Delta\bigr)^2} \\ &=\; \mu^{4-D} \int \frac{d^Dq_E}{(2\pi)^D}\, \frac{-\frac{4}{D}q_E^2 \;+\; (1-2x)^2 k^2}{(q_E^2+\Delta)^2} \\ &=\; \mu^{4-D} \int_0^\infty dt\, t \int \frac{d^Dq_E}{(2\pi)^D}\, \bigl(-\tfrac{4}{D}q_E^2 \;+\; (1-2x)^2 k^2\bigr) e^{-t(q_E^2+\Delta)} \end{aligned}$$

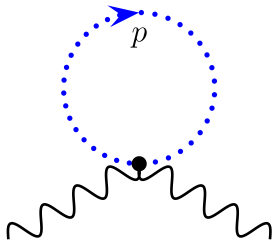
$$
\begin{aligned}
&= \mu^{4-D}\int_0^\infty dt\, t e^{-t\Delta}\left(+\frac{4}{D}\frac{\partial}{\partial t} + (1-2x)^2k^2\right)\int\frac{d^Dq_E}{(2\pi)^D}\,e^{-tq_E^2} \\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\int_0^\infty dt\, t e^{-t\Delta}\left(\frac{4}{D}\frac{\partial}{\partial t} + (1-2x)^2k^2\right)t^{-D/2} \\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\int_0^\infty dt\, t e^{-t\Delta}\left(-2t^{-(D/2)-1} + (1-2x)^2k^2\times t^{-D/2}\right) \\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\left(-2\Gamma\left(1-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-1} + (1-2x)^2k^2\times\Gamma\left(2-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-2}\right) \\
&\qquad \langle\!\langle \text{ using } \Gamma(2-\tfrac{D}{2}) = (1-\tfrac{D}{2})\Gamma(1-\tfrac{D}{2}) \text{ and } \tfrac{\partial\Delta}{\partial x} = (2x-1)k^2 \,\rangle\!\rangle \\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\Gamma\left(1-\tfrac{D}{2}\right)\times\left[-2\Delta^{\frac{D}{2}-1} + \left(\tfrac{D}{2}-1\right)\Delta^{\frac{D}{2}-2}\times(1-2x)\,\frac{\partial\Delta}{\partial x}\right] \\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(1-\tfrac{D}{2}\right)\times\frac{\partial}{\partial x}\left((1-2x)\Delta^{\frac{D}{2}-1}\right), \qquad \text{(S.12)}
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\Xi_{(1)}(k^2) &= e^2\int_0^1 dx\,\frac{\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(1-\tfrac{D}{2}\right)\,\frac{\partial}{\partial x}\left((1-2x)\Delta^{\frac{D}{2}-1}\right) \\
&= \frac{e^2\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(1-\tfrac{D}{2}\right)\times\left[-\,\Delta^{\frac{D}{2}-1}\Big|_{x=1} - \Delta^{\frac{D}{2}-1}\Big|_{x=0} = -2(M^2)^{\frac{D}{2}-1}\right] \qquad \text{(S.13)} \\
&= -\frac{\alpha M^2}{2\pi}\times\Gamma\left(1-\tfrac{D}{2}\right)\times\left(\frac{4\pi\mu^2}{M^2}\right)^{2-\frac{D}{2}}.
\end{aligned}
$$

Note that thanks to $\Delta(x=1) = \Delta(x=0) = M^2$, the bottom line of eq. (S.13) is independent of the photon's momentum k. And since the second diagram's contribution $\Sigma^{\mu\nu}_{(2)}$ is also k–independent, this allows for the cancellation of the wrong tensor structure of the two-photon amplitude between the two diagrams.

Indeed, for the second diagram (2)



(S.14)

we have

$$i\Sigma^{\mu\nu}_{(2)} \;=\; \int\frac{d^4p}{(2\pi)^4}\, 2ie^2 g^{\mu\nu} \times \frac{i}{p^2 - M^2 + i0} \tag{S.15}$$

which does not depend on the photons' momenta and has wrong tensor structure

$$\Sigma^{\mu\nu}_{(2)} \;=\; g^{\mu\nu} \times \Xi_{(2)}\,. \tag{S.16}$$

To evaluate the coefficient $\Xi_{(2)}$ of this wrong tensor structure, we continue the loop momentum p to Euclidean space and then use dimensional regularization, thus

$$\begin{aligned}
\Xi_{(2)} \;&=\; \int\limits_{\text{reg}}\frac{d^4p}{(2\pi)^4}\,\frac{2ie^2}{p^2 - M^2 + i0} \\
&=\; \int\limits_{\text{reg}}\frac{d^4p_E}{(2\pi)^4}\,\frac{-2e^2}{-(p_E^2 + M^2)} \\
&=\; 2e^2\mu^{4-D}\int\frac{d^Dp_E}{(2\pi)^D}\int\limits_0^\infty dt\, e^{-t(M^2+p_E^2)} \\
&=\; 2e^2\mu^{4-D}\int\limits_0^\infty dt\, e^{-tM^2}\int\frac{d^Dp_E}{(2\pi)^D} e^{-tp_E^2} \\
&=\; \frac{2e^2\mu^{4-D}}{(4\pi)^{D/2}}\int\limits_0^\infty dt\, e^{-tM^2} t^{-D/2} \\
&=\; \frac{2e^2\mu^{4-D}}{(4\pi)^{D/2}} \times \Gamma\left(1 - \tfrac{D}{2}\right)\, (M^2)^{\frac{D}{2}-1} \\
&=\; \frac{\alpha M^2}{2\pi} \times \Gamma\left(1 - \tfrac{D}{2}\right) \times \left(\frac{4\pi\mu^2}{M^2}\right)^{2-\frac{D}{2}}
\end{aligned} \tag{S.17}$$

Comparing this formula to the wrong-tensor contribution (S.13) of the first diagram, we immediately see that they cancel each other,

$$\Xi_{1\,\text{loop}} \;=\; \Xi_{(1)} \;+\; \Xi_{(2)} \;=\; 0 \tag{S.18}$$

and therefore

$$\Sigma^{\mu\nu}_{1\,\text{loop}} \;=\; \left(k^\mu k^\nu \;-\; k^2 g^{\mu\nu}\right) \times \Pi_{1\,\text{loop}}(k^2) \tag{2}$$

$\mathcal{Q.E.D.}$

Alternative solution:

Above, we had to integrate over the Feynman parameter x to see that the wrong-tensor contribution from the two diagrams cancel each other. Alternatively, we may combine the two diagrams together before integrating over momenta or Feynman parameters. We can do that by identifying the loop momentum p of the second diagram with either p or $p' = p+k$ of the first diagram. For symmetry's sake, let's take the average of the two identifications, thus

$$\left[\frac{1}{p^2 - M^2 + i0}\right]_{(2)} \;\to\; \frac{1/2}{p^2 - M^2 + i0} + \frac{1/2}{p'^2 - M^2 + i0} \;=\; \frac{\frac{1}{2}p^2 + \frac{1}{2}p'^2 - M^2}{(p^2 - M^2 + i0)(p'^2 - M^2 + i0)}\,. \tag{S.19}$$

This trick brings both diagrams to a common denominator, so adding them up yields

$$i\Sigma^{\mu\nu}_{(1+2)} \;=\; e^2 \int\!\frac{d^4p}{(2\pi)^4}\,\frac{(p+p')^\mu(p+p')^\nu - g^{\mu\nu}(p^2 + p'^2 - 2M^2)}{(p^2 - M^2 + i0) \times (p'^2 - M^2 + i0)}\,. \tag{S.20}$$

At this point, we may simplify the common denominator using the Feynman's parameter trick. Proceeding exactly as in eqs. (S.3)–(S.5), we have

$$\Sigma^{\mu\nu}_{(1+2)} \;=\; -ie^2 \int\limits_0^1\! dx \int\limits_{\text{reg}}\! \frac{d^4q}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}}{[q^2 - \Delta + i0]^2} \tag{S.21}$$

where q and Δ are exactly as in eq. (S.5), while in the numerator

$$\begin{aligned}
(p+p')^\mu(p+p')^\nu \;&=\; (2q + (1-2x)k)^\mu (2q + (1-2x)k)^\nu \\
&=\; 4q^\mu q^\nu \;+\; (1-2x)^2 k^\mu k^\nu \;+\; 2(1-2x)(q^\mu k^\nu + k^\mu q^\nu) \\
&\qquad \langle\!\langle \text{ disregarding odd powers of } q \text{ which integrate to zero } \rangle\!\rangle \\
&\cong\; 4q^\mu q^\nu \;+\; (1-2x)^2 k^\mu k^\nu,
\end{aligned} \tag{S.22}$$

$$
\begin{aligned}
p^2 \;+\; p'^2 \;-\; 2M^2 &= \;(q-xk)^2 \;+\; (q+(1-x)k)^2 \;-\; 2\Delta -\; 2x(1-x)k^2 \\
&= \;2q^2 \;+\; (x^2+(1-x)^2)k^2 \;+\; 2(1-2x)(kq) \;-\; 2\Delta -\; 2x(1-x)k^2 \\
&\quad \langle\!\langle \text{ Disregarding odd powers of } q \,\rangle\!\rangle \\
&\cong 2q^2 \;+\; k^2\bigl(x^2+(1-x)^2-2x(1-x) \;=\; (1-2x)^2\bigr) \;-\; 2\Delta,
\end{aligned}
\tag{S.23}
$$

hence

$$
\mathcal{N}^{\mu\nu} \;\cong\; (1-2x)^2\times(k^\mu k^\nu - g^{\mu\nu}k^2) \;+\; 4q^\mu q^\nu \;+\; g^{\mu\nu}(2\Delta-2q^2). \tag{S.24}
$$

Furthermore, in the context of dimensionally regulated integral (S.21)

$$
q^\mu q^\nu \;\cong\; \frac{q^2}{D}\times g^{\mu\nu}, \tag{S.25}
$$

hence

$$
\mathcal{N}^{\mu\nu} \;\cong\; (1-2x)^2\times(k^\mu k^\nu - g^{\mu\nu}k^2) \;+\; g^{\mu\nu}\times\left(\frac{4}{D}\,q^2 \;-\; 2q^2 \;+\; 2\Delta\right) \tag{S.26}
$$

and therefore

$$
\Sigma^{\mu\nu}_{(1+2)}(k) \;=\; \bigl(k^2 g^{\mu\nu} \;-\; k^\mu k^\nu\bigr)\times\Pi_{(1+2)}(k^2) \;+\; g^{\mu\nu}\times\Xi_{(1+2)}(k^2) \tag{S.27}
$$

where

$$
\Pi_{(1+2)}(k^2) \;=\; ie^2\int_0^1\! dx\,(1-2x)^2\int_{\text{reg}}\frac{d^4q}{(2\pi)^4}\,\frac{1}{\bigl(q^2-\Delta+i0\bigr)^2} \tag{S.28}
$$

and

$$
\Xi_{(1+2)}(k^2) \;=\; -ie^2\int_0^1\! dx\int_{\text{reg}}\frac{d^4q}{(2\pi)^4}\,\frac{2\Delta \;-\; (2-\frac{4}{D})q^2}{(q^2-\Delta+i0)^2}\,. \tag{S.29}
$$

The wrong-tensor contribution (S.29) for the scalar loop looks exactly like the similar wrong-tensor contribution of the electron loop we have seen in class, and the momentum integral here vanishes in exactly the same way. Using dimensional regularization, we analytically continue to

$\operatorname{Re} D < 2$ (because the integral diverges quadratically in $D = 4$), and them we get zero for all D. Indeed,

$$\begin{aligned}
-i\int\frac{d^Dq}{(2\pi)^D}\,\frac{2\Delta\,-\,(2-\frac{4}{D})q^2}{(q^2-\Delta+i0)^2}\ &=\ \int\frac{d^Dq_E}{(2\pi)^D}\,\frac{2\Delta+(2-\frac{4}{D})q_E^2}{(q_E^2+\Delta)^2}\\
&=\int\limits_0^\infty dt\,t\times e^{-\Delta t}\times\left(2\Delta\,-\,(2-\tfrac{4}{D})\frac{\partial}{\partial t}\right)\int\frac{d^Dq_E}{(2\pi)^D}\,e^{-tq_E^2}\\
&=\int\limits_0^\infty dt\,t\times e^{-\Delta t}\times\left(2\Delta\,-\,(2-\tfrac{4}{D})\frac{\partial}{\partial t}\right)(4\pi t)^{-D/2}\\
&\propto\int\limits_0^\infty dt\,e^{-\Delta t}\times\left(2\Delta\times t^{1-(D/2)}\,+\,(D-2)\times t^{-D/2}\right)\\
&=\ 2\Delta\times\Gamma(2-\tfrac{D}{2})\Delta^{2-(D/2)}\ +\ (D-2)\times\Gamma(1-\tfrac{D}{2})\Delta^{1-(D/2)}\\
&\qquad\langle\!\langle\text{using }\Gamma(2-\tfrac{D}{2})=\Gamma(1-\tfrac{D}{2})\times(1-\tfrac{D}{2})\rangle\!\rangle\\
&=\ \Delta^{1-(D/2)}\Gamma(1-\tfrac{D}{2})\times\left[2(1-\tfrac{D}{2})\,+\,(D-2)\ =\ 0\right]\\
&=\ 0\ \forall D<2.
\end{aligned}\tag{S.30}$$

Analytically continuing this formula back to $D = 4$ we have $\Xi_{1+2}(k^2) \equiv 0$ and hence $\Sigma^{\mu\nu}_{(1+2)}(k) = (g^{\mu\nu}k^2 - k^\mu k^\nu) \times \Pi_{1+2}(k^2)$ as in eq. (1), *quod erat demonstrandum.*

Problem **2**(b):

Our next task is to calculate the $\Pi_{1\,\mathrm{loop}}(k^2)$ factor in eq. (2). As we saw in part (a), only the first diagram contributes to the correct tensor structure in $\Sigma^{\mu\nu}_{1\,\mathrm{loop}}$, hence according to eq. (S.10)

$$\Pi_{1\,\mathrm{loop}}(k^2)\ =\ ie^2\int\limits_0^1 dx\,(1-2x)^2\int\limits_{\mathrm{reg}}\frac{d^4q}{(2\pi)^4}\,\frac{1}{\left(q^2-\Delta+i0\right)^2}\tag{S.31}$$

In the alternative solution, eq. (S.28) gives exactly the same formula for the $\Pi_{(1+2)}(k^2)$ for the net contribution of the two diagrams.

The momentum integral in eq. (S.31) should be rather familiar after so much related class-work and home-work, so let me simply write down the result:

$$i\int\limits_{\rm reg}\frac{d^4q}{(2\pi)^4}\,\frac{1}{\bigl(q^2-\Delta+i0\bigr)^2}\ =\ \frac{-1}{16\pi^2}\,\Gamma\left(2-\tfrac{D}{2}\right)\,\left(\frac{4\pi\mu^2}{\Delta}\right)^{2-\frac{D}{2}}\ \xrightarrow[D\to4]{}\ \frac{-1}{16\pi^2}\left(\frac{1}{\epsilon}\ -\ \gamma_E\ +\ \log\frac{4\pi\mu^2}{\Delta}\right). \tag{S.32}$$

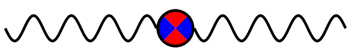
Consequently,

$$\begin{aligned}\Pi_{1\,\rm loop}(k^2)\ &=\ -\frac{\alpha}{4\pi}\int\limits_0^1 dx\,(1-2x)^2\,\left(\frac{1}{\epsilon}\ -\ \gamma_E\ +\ \log\frac{4\pi\mu^2}{M^2}\ +\ \log\frac{M^2}{\Delta(x)}\right)\\ &=\ -\frac{\alpha}{12\pi}\left(\frac{1}{\epsilon}\ -\ \gamma_E\ +\ \log\frac{4\pi\mu^2}{M^2}\ +\ \widehat{I}(k^2/M^2)\right)\end{aligned} \tag{S.33}$$

where

$$\widehat{I}(k^2/M^2)\ =\ 3\int\limits_0^1 dx\,(1-2x)^2\log\frac{M^2}{\Delta(x)=M^2-x(1-x)k^2}\ =\ -3J(k^2/M^2)-2I(k^2/M^2). \tag{S.34}$$

Finally, note that eq. (S.33) is the ampitude due to one-loop diagrams (1) themselves, without accounting for the counterterm contribution



Adding the counterterm δ_3 to the picture, we get

$$\Sigma^{\mu\nu}_{\rm net}(k)\ =\ \Sigma^{\mu\nu}_{\rm loops}(k)\ -\ \delta_3(k^2g^{\mu\nu}-k^\mu k^\nu) \tag{S.35}$$

and hence

$$\Pi_{\rm net}(k^2)\ =\ \Pi_{\rm loops}(k^2)\ -\ \delta_3\,. \tag{S.36}$$

The renormalization condition for the δ_3 is $\Pi_{\rm net}=0$ for $k^2=0$; this assures that the dressed propagator for the renormalized EM field has a pole at $k^2=0$ with residue $=1$. Consequently

$$\delta_3 \;=\; \Pi_{\text{loops}}(k^2=0), \tag{S.37}$$

hence given the one-loop formula (S.33) where $\widehat{I}(0)=0$ (*cf.* eq. (S.34)), the counterterm is

$$\Delta_3 \;=\; -\frac{\alpha}{12\pi}\left(\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{M^2}\right)\,+\,O(\alpha^2/\pi^2), \tag{S.38}$$

and the net propagator correction is

$$\Pi_{\text{net}}(k^2) \;=\; -\frac{\alpha}{12\pi}\,\widehat{I}(k^2/M^2)\;+\;O(\alpha^2/\pi^2). \tag{S.39}$$

Problem **2**(c):

At high momenta $k^2 \gg M^2$, we may approximate

$$\log\frac{M^2}{M^2-x(1-x)k^2}\;\approx\;\log\frac{M^2}{-x(1-x)k^2}\;=\;-\log\frac{-k^2}{M^2}\;+\;\log\frac{1}{x(1-x)} \tag{S.40}$$

and hence

$$\widehat{I}(k^2/M^2)\;\approx\;3\int_0^1\!dx\,(1-2x)^2\left[-\log\frac{-k^2}{M^2}\;+\;\log\frac{1}{x(1-x)}\right]\;=\;-\log\frac{-k^2}{M^2}\;+\;\frac{8}{3}\,. \tag{S.41}$$

Consequently, at high momenta the "vacuum polarization" factor $\Pi(k^2)$ behaves as

$$\Pi(k^2)\;=\;\frac{\alpha}{12\pi}\left(+\log\frac{-k^2}{M^2}\;-\;\frac{8}{3}\;+\;O(M^2/k^2)\right)\;+\;O(\alpha^2), \tag{S.42}$$

and therefore the effective gauge coupling

$$\alpha_{\text{eff}}(k^2)\;=\;\frac{\alpha}{1-\Pi(k^2)} \tag{S.43}$$

behaves according to

$$\frac{1}{\alpha_{\text{eff}}(k^2)}\;\approx\;\frac{1}{\alpha(0)}\;-\;\frac{1}{12\pi}\left(\log\frac{-k^2}{M^2}\;-\;\frac{8}{3}\right). \tag{3}$$

Quod erat demonstrandum.