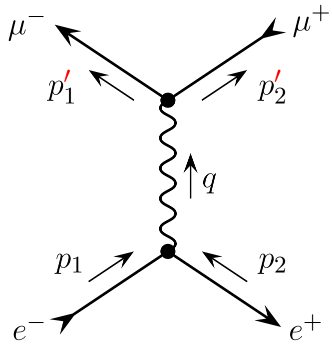
Muon Pair Productions and Dirac Trace Techniques

Consider an $e^- + e^+$ collision in which a muon pair $\mu^- + \mu^+$ is created. There is one tree diagram for this process,



(1)

and it evaluates to

$$i\left\langle \mu^-, \mu^+ \right| \mathcal{M} \left| e^-, e^+ \right\rangle = \frac{-ig^{\lambda\nu}}{q^2} \times \bar{u}(\mu^-)(ie\gamma_\lambda)v(\mu^+) \times \bar{v}(e^+)(ie\gamma_\nu)u(e^-)$$
$$= \frac{ie^2}{s} \times \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \times \bar{v}(e^+)\gamma_\nu u(e^-) \tag{2}$$

where

$$s = q^2 = (p_1 + p_2)^2 = (p_1' + p_2')^2 = E^2_{\text{c.m.}} \tag{3}$$

is the square of the total energy in the center-of-mass frame.

The amplitude (2) depends on the spins of all 4 particles involved. In the homework set#10 (problem 2), you should calculate such *polarized* cross-sections for specific helicities of all the particles in the ultra-relativistic limit. But in this set of notes we shall focus on the *unpolarized* cross-sections. That is, envision an experiment in which the initial electron and positron beams include particles with both spin states in a 50%/50% mix. Moreover, the muon detector is sensitive to the final-state muon's momenta but is blind to their spin states. Consequently, the un-polarized partial cross-section measured in such experiment is the sum/average of the polarized partial cross-sections: Sum over the final muon spin states and average over the spin states of the initial electron and positron. Thus,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpolarized}} = \frac{1}{4}\sum_{s_1,s_2}\sum_{s_1',s_2'}\left(\frac{d\sigma}{d\Omega}\right)_{\text{polarized}} . \tag{4}$$

Or in terms of the polarized amplitudes (2),

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{|\mathbf{p}'|}{|\mathbf{p}|}\,\frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} \quad\text{where}\quad \overline{|\mathcal{M}|^2} \stackrel{\text{def}}{=} \frac{1}{4}\sum_{s_1,s_2,s_1',s_2'}|\mathcal{M}|^2. \tag{5}$$

Unpolarized cross-sections are quite common in high energy physics, so we should learn how to sum or average the amplitudes — or rather mod-squared amplitudes — over the particle's spin states. In these notes, we shall focus on the Dirac trace techniques for summing or averaging over spins of Dirac fermions like electrons or muons.

Spin Sums as Dirac Traces

Let's put the muon pair production aside for a moment while we work out a simpler example of spin sum/average. Consider some QED process involving one incoming electron with momentum p and spin s, one outgoing electron with momentum p' and spin s', and some photons, incoming or outgoing. There may be several Feynman diagrams contributing to the amplitude for this process, but all such diagrams have the same external electron legs and the corresponding factors $u(p,s)$ and $\bar{u}(p',s')$. Consequently, we may write the amplitude as

$$\left\langle e^{-\prime},\ldots\right|\mathcal{M}\left|e^-,\ldots\right\rangle = \bar{u}(p',s')\Gamma u(p,s) \tag{6}$$

where Γ is comprises all the other factors of the QED Feynman rules; for the moment, we don't want to be specific, so Γ is just some kind of a 4×4 matrix. In general, Γ depends on all particles' momenta as well as external photons' polarizations, but it does not depend on the spin states s and s' of the incoming and the outgoing electrons. Instead, the amplitude (6) depends on s and s' *only* through the $u(p,s)$ and $\bar{u}(p',s')$ factors.

In many experiments, the initial electrons come in un-polarized beam, 50% having spin $s=+\frac{1}{2}$ and 50% having $s=-\frac{1}{2}$. At the same time, the detector for the final electrons measures their momenta p' but is blind to their spins s'. The cross-section $\overline{\sigma}$ measured by such an experiment would be the average of the polarized cross-sections $\sigma(s,s')$ with respect to initial spins s and the sum over the final spins s', thus

$$\overline{\sigma} \;=\; \frac{1}{2}\sum_{s}\sum_{s'}\sigma(s,s'). \tag{7}$$

Similar averaging / summing rules apply to the un-polarized partial cross-sections,

$$\overline{\frac{d\sigma}{d\Omega}} \;=\; \frac{1}{2}\sum_{s}\sum_{s'}\frac{d\sigma(s,s')}{d\Omega}\,, \tag{8}$$

etc., etc. Since all total or partial cross-sections are proportional to mod-squares $|\mathcal{M}|^2$ of amplitudes $\mathcal{M}$, we need to know how to calculate

$$\overline{|\mathcal{M}|^2} \;\stackrel{\text{def}}{=}\; \frac{1}{2}\sum_{s}\sum_{s'}\bigl|\mathcal{M}(s,s')\bigr|^2 \tag{9}$$

for amplitudes such as (6).

To do perform such calculations efficiently, we need to recall two things about Dirac spinors. First,

$$\textbf{If}\quad \mathcal{M} \;=\; \bar{u}(p',s')\Gamma u(p,s) \qquad \textbf{then}\quad \mathcal{M}^* \;=\; \bar{u}(p,s)\overline{\Gamma}u(p',s') \tag{10}$$

where $\overline{\Gamma}=\gamma^0\Gamma^\dagger\gamma^0$ is the Dirac conjugate of the matrix Γ; for a product $\gamma_\lambda\cdots\gamma_\nu$ of Dirac matrices, $\overline{\gamma_\lambda\cdots\gamma_\nu}=\gamma_\nu\cdots\gamma_\lambda$. Second,

$$\sum_{s}u_\alpha(p,s)\times\bar{u}_\beta(p,s) \;=\; (\not{p}+m)_{\alpha\beta} \tag{11}$$

and likewise

$$\sum_{s'}u_\gamma(p',s')\times\bar{u}_\delta(p',s') \;=\; (\not{p}'+m)_{\gamma\delta}\,. \tag{12}$$

Combining these two facts, we obtain

$$\begin{aligned}
\sum_{s,s'} |\mathcal{M} = \bar{u}(p',s')\Gamma u(p,s)|^2 &= \sum_{s,s'} \bar{u}(p',s')\Gamma u(p,s) \times \bar{u}(p,s)\overline{\Gamma} u(p',s') \\
&= \sum_{s,s'} \left(\sum_{\delta,\alpha} \bar{u}_\delta(p',s')\,\Gamma_{\delta\alpha}\,u_\alpha(p,s) \times \sum_{\beta,\gamma} \bar{u}_\beta(p,s)\,\overline{\Gamma}_{\beta\gamma}\,u_\gamma(p',s') \right) \\
&= \sum_{\alpha,\beta,\gamma,\delta} \Gamma_{\delta\alpha}\,\overline{\Gamma}_{\beta\gamma} \times \left(\sum_s u_\alpha(p,s)\,\bar{u}_\beta(p,s) \right) \\
&\qquad\qquad \times \left(\sum_{s'} u_\gamma(p',s')\,\bar{u}_\delta(p',s') \right) \\
&= \sum_{\alpha,\beta,\gamma,\delta} \Gamma_{\delta\alpha}\,\overline{\Gamma}_{\beta\gamma} \times (\not{p}+m)_{\alpha\beta} \times (\not{p}'+m)_{\gamma\delta} \\
&= \sum_\gamma \Big(\text{matrix product } (\not{p}'+m)\,\Gamma\,(\not{p}+m)\,\overline{\Gamma} \Big)_{\gamma\gamma} \\
&= \operatorname{tr}\Big((\not{p}'+m)\,\Gamma\,(\not{p}+m)\,\overline{\Gamma} \Big)
\end{aligned} \tag{13}$$

and hence

$$\left\langle e^{-\prime},\ldots\right| \mathcal{M} \left|e^-,\ldots\right\rangle = \bar{u}(p',s')\Gamma u(p,s) \quad\Longrightarrow\quad \frac{1}{2}\sum_{s,s'} |\mathcal{M}|^2 = \frac{1}{2}\operatorname{tr}\Big((\not{p}'+m)\,\Gamma\,(\not{p}+m)\,\overline{\Gamma} \Big). \tag{14}$$

A similar trace formula exists for un-polarized scattering of positrons. In this case, the amplitude is

$$\left\langle e^{+\prime},\ldots\right| \mathcal{M} \left|e^+,\ldots\right\rangle = \bar{v}(p,s)\Gamma v(p',s') \tag{15}$$

(note that $v(p',s')$ belongs to the outgoing positron while $\bar{v}(p,s)$ belongs to the incoming s^+), and we need to average $|\mathcal{M}|^2$ over s and sum over s'. Using

$$\sum_s v_\alpha(p,s) \times \bar{v}_\beta(p,s) = (\not{p}-m)_{\alpha\beta} \tag{16}$$

and working through algebra similar to eq. (13), we arrive at

$$\left\langle e^{+\prime},\ldots\right| \mathcal{M} \left|e^+,\ldots\right\rangle = \bar{v}(p,s)\Gamma v(p',s') \quad\Longrightarrow\quad \frac{1}{2}\sum_{s,s'} |\mathcal{M}|^2 = \frac{1}{2}\operatorname{tr}\Big((\not{p}-m)\,\Gamma\,(\not{p}'-m)\,\overline{\Gamma} \Big). \tag{17}$$

Now suppose an electron with momentum p_1 and spin s_1 and a positron with momentum p_2 and spin s_2 come in and annihilate each other. In this case, the amplitude has form

$$\langle\ldots|\,\mathcal{M}\,|e_1^-,e_2^+,\ldots\rangle \;=\; \bar{v}(p_2,s_2)\Gamma u(p_1,s_1) \tag{18}$$

for some Γ, and if both electron and positron beams are un-polarized, we need to average the $|\mathcal{M}|^2$ over both spins s_1 and s_2. Again, there is a trace formula for such averaging, namely

$$\langle\ldots|\,\mathcal{M}\,|e_1^-,e_2^+,\ldots\rangle \;=\; \bar{v}(p_2,s_2)\Gamma u(p_1,s_1) \quad\Longrightarrow\quad \frac{1}{4}\sum_{s_1,s_2}|\mathcal{M}|^2 \;=\; \frac{1}{4}\,\mathrm{tr}\Big((\not{p}_2-m)\,\Gamma\,(\not{p}_1+m)\,\overline{\Gamma}\Big). \tag{19}$$

Likewise, for a process in which an electron-positron pair is created, the amplitude has form

$$\left\langle e_1^{-\prime},e_2^{+\prime},\ldots\right|\mathcal{M}\,|\ldots\rangle \;=\; \bar{u}(p_1',s_1')\Gamma v(p_2',s_2'), \tag{20}$$

and if we do not detect the spins of the outgoing electron and positron but only their momenta, then we should sum the $|\mathcal{M}|^2$ over both spins s_1 and s_2. Again, there is a trace formula for this sum, namely

$$\left\langle e_1^{-\prime},e_2^{+\prime},\ldots\right|\mathcal{M}\,|\ldots\rangle \;=\; \bar{u}(p_1',s_1')\Gamma v(p_2',s_2') \quad\Longrightarrow\quad \sum_{s_1,s_2}|\mathcal{M}|^2 \;=\; \mathrm{tr}\Big((\not{p}_1'+m)\,\Gamma\,(\not{p}_2'-m)\,\overline{\Gamma}\Big). \tag{21}$$

There are similar Dirac trace formulæ for the processes involving 4 or more un-polarized fermions. For example, let's go back to muon pair production in electron-positron collisions, $e^-+e^+\to\mu^-+\mu^+$. The tree-level amplitude for this process is

$$\mathcal{M} \;=\; \frac{e^2}{s}\times\bar{u}(\mu^-)\gamma^\nu v(\mu^+)\times\bar{v}(e^+)\gamma_\nu u(e^-), \tag{2}$$

and the un-polarized partial cross-section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{c.m.}} \;=\; \frac{|\mathbf{p}'|}{|\mathbf{p}|}\,\frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} \quad\text{where}\quad \overline{|\mathcal{M}|^2} \;=\; \frac{1}{4}\sum_{s_1,s_2,s_1',s_2'}|\mathcal{M}|^2. \tag{22}$$

The complex conjugate of the amplitude (2) is

$$\mathcal{M}^* \;=\; \frac{e^2}{s}\times\bar{v}(\mu^+)\gamma^\lambda u(\mu^-)\times\bar{u}(e^-)\gamma_\lambda v(e^+), \tag{23}$$

where I changed the summation Lorentz index from ν to λ so I can multiply eqs. (23) and

(2) together. Consequently,

$$\begin{aligned}\mathcal{M}\times\mathcal{M}^* &= \frac{e^4}{s^2}\times\Big(\bar{u}(\mu^-)\gamma^\nu v(\mu^+)\times\bar{v}(e^+)\gamma_\nu u(e^-)\Big)\times\Big(\bar{v}(\mu^+)\gamma^\lambda u(\mu^-)\times\bar{u}(e^-)\gamma_\lambda v(e^+)\Big)\\ &= \frac{e^4}{s^2}\times\Big(\bar{u}(\mu^-)\gamma^\nu v(\mu^+)\times\bar{v}(\mu^+)\gamma^\lambda u(\mu^-)\Big)\times\Big(\bar{v}(e^+)\gamma_\nu u(e^-)\times\bar{u}(e^-)\gamma_\lambda v(e^+)\Big),\end{aligned} \tag{24}$$

and when we spin sum/average this amplitude mod-square, we get

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{e^4}{4s^2}\times\left(\sum_{s'_1,s'_2}\bar{u}(\mu^-)\gamma^\nu v(\mu^+)\times\bar{v}(\mu^+)\gamma^\lambda u(\mu^-)\right)\times\\ &\qquad\times\left(\sum_{s_1,s_2}\bar{v}(e^+)\gamma_\nu u(e^-)\times\bar{u}(e^-)\gamma_\lambda v(e^+)\right)\\ &= \frac{e^4}{4s^2}\times\mathrm{tr}\Big((\not{p}'_1+M_\mu)\,\gamma^\nu\,(\not{p}'_2-M_\mu)\,\gamma^\lambda\Big)\times\mathrm{tr}\Big((\not{p}_2-m_e)\,\gamma_\nu\,(\not{p}_1+m_e)\,\gamma_\lambda\Big).\end{aligned} \tag{25}$$

Calculating Dirac Traces

Thus far, we have learned how to express un-polarized cross-sections in terms of Dirac traces (*i.e.*, traces of products of the Dirac γ^λ matrices). In this section, we shall learn how to calculate such traces.

Dirac traces do not depend on the specific form of the $\gamma^0,\gamma^1,\gamma^2,\gamma^4$ matrices but are completely determined by the Clifford algebra

$$\{\gamma^\mu,\gamma^\nu\}\;\equiv\;\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\;=\;2g^{\mu\nu}. \tag{26}$$

To see how this works, please recall the key property of the trace of any matrix product: $\mathrm{tr}(AB)=\mathrm{tr}(BA)$ for any two matrices A and B. This symmetry has two important corollaries:

- All commutators have zero traces, $\mathrm{tr}([A,B])=0$ for any A and B.

- Traces of products of several matrices have *cyclic symmetry*

$$\mathrm{tr}(ABC\cdots YZ) \;=\; \mathrm{tr}(BC\cdots YZA) \;=\; \mathrm{tr}(C\cdots YZAB) \;=\; \cdots \;=\; \mathrm{tr}(ZABC\cdots Y). \tag{27}$$

Using these properties it is easy to show that

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) \;=\; 4g^{\mu\nu} \quad\Longrightarrow\quad \mathrm{tr}(\not{a}\,\not{b}) \;=\; 4(ab) \;\equiv\; 4a_\mu b^\mu. \tag{28}$$

Indeed,

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) \;=\; \mathrm{tr}(\gamma^\nu\gamma^\mu) \;=\; \mathrm{tr}\left(\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\right) \;=\; \mathrm{tr}(g^{\mu\nu}) \;=\; g^{\mu\nu}\times\mathrm{tr}(1) \;=\; g^{\mu\nu}\times 4, \tag{29}$$

where the last equality follows from Dirac matrices being 4×4 and hence

$$\mathrm{tr}(1) \;=\; 4. \tag{30}$$

Next, all products of any *odd* numbers of the γ^μ matrices have zero traces,

$$\mathrm{tr}(\gamma^\mu) \;=\; 0, \quad \mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu) \;=\; 0, \quad \mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \;=\; 0,\ \textit{etc.}, \tag{31}$$

and hence

$$\mathrm{tr}(\not{a}) \;=\; 0, \quad \mathrm{tr}(\not{a}\,\not{b}\,\not{c}) \;=\; 0, \quad \mathrm{tr}(\not{a}\,\not{b}\,\not{c}\,\not{d}\,\not{e}) \;=\; 0,\ \textit{etc.} \tag{32}$$

To see how this works, we can use the γ^5 matrix which anticommutes with all the γ^μ and hence with any product Γ of an odd number of the Dirac γ's, $\gamma^5\Gamma = -\Gamma\gamma^5$. Combining this observation with $\gamma^5\gamma^5 = 1$, we have

$$\Gamma \;=\; \gamma^5\gamma^5\Gamma \;=\; -\gamma^5\Gamma\gamma^5 \;=\; -\tfrac{1}{2}[\gamma^5\Gamma,\gamma^5] \tag{33}$$

and therefore

$$\mathrm{tr}(\Gamma) \;=\; -\tfrac{1}{2}\,\mathrm{tr}([\gamma^5\Gamma,\gamma^5]) \;=\; 0. \tag{34}$$

Products of even numbers $n = 2m$ of γ matrices have non-trivial traces, and we may calculate them recursively in n. We already know the traces for $n = 0$ and $n = 2$, so consider a product $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ of $n=4$ matrices. Thanks to the cyclic symmetry of the trace,

$$\mathrm{tr}(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu) \;=\; \mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\kappa) \;=\; \mathrm{tr}\left(\tfrac{1}{2}\{\gamma^\kappa,\gamma^\lambda\gamma^\mu\gamma^\nu\}\right) \tag{35}$$

where the anticommutator follows from the Clifford algebra (26) and the Leibniz rule,

$$\begin{aligned}\{\gamma^\kappa,\gamma^\lambda\gamma^\mu\gamma^\nu\} \;&=\; \{\gamma^\kappa,\gamma^\lambda\}\gamma^\mu\gamma^\nu \;-\; \gamma^\lambda\{\gamma^\kappa,\gamma^\mu\}\gamma^\nu \;+\; \gamma^\lambda\gamma^\mu\{\gamma^\kappa,\gamma^\nu\}\\ &=\; 2g^{\kappa\lambda}\times\gamma^\mu\gamma^\nu \;-\; 2g^{\kappa\mu}\times\gamma^\lambda\gamma^\nu \;+\; 2g^{\kappa\nu}\times\gamma^\lambda\gamma^\mu.\end{aligned} \tag{36}$$

Consequently

$$\begin{aligned}\mathrm{tr}(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu) \;&=\; g^{\kappa\lambda}\times\mathrm{tr}(\gamma^\mu\gamma^\nu) \;-\; g^{\kappa\mu}\times\mathrm{tr}(\gamma^\lambda\gamma^\nu) \;+\; g^{\kappa\nu}\times\mathrm{tr}(\gamma^\lambda\gamma^\mu)\\ &=\; 4g^{\kappa\lambda}g^{\mu\nu} \;-\; 4g^{\kappa\mu}g^{\lambda\nu} \;+\; 4g^{\kappa\nu}g^{\lambda\mu}\end{aligned} \tag{37}$$

and hence

$$\mathrm{tr}(\not{a}\,\not{b}\,\not{c}\,\not{d}) \;=\; 4(ab)(cd) \;-\; 4(ac)(bd) \;+\; 4(ad)(bc). \tag{38}$$

Note that in eq. (37) we have expressed the trace of a 4–γ product to traces of 2–γ products. Similar recursive formulae exist for all even numbers of γ matrices,

$$\begin{aligned}\mathrm{tr}(\gamma^{\nu_1}\gamma^{\nu_2}\cdots\gamma^{\nu_n}) \;&=\; \mathrm{tr}\left(\tfrac{1}{2}\{\gamma^{\nu_1},\gamma^{\nu_2}\cdots\gamma^{\nu_n}\}\right)\\ &\qquad\text{for even } n \text{ only}\\ &=\; \sum_{k=2}^{n}(-1)^k g^{\nu_1\nu_k}\times\mathrm{tr}\left(\gamma^{\nu_2}\cdots\cancel{\gamma^{\nu_k}}\cdots\gamma^{\nu_n}\right).\end{aligned} \tag{39}$$

For example, for $n=6$

$$\begin{aligned}\mathrm{tr}(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \;=\; & g^{\kappa\lambda}\times\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \;-\; g^{\kappa\mu}\times\mathrm{tr}(\gamma^\lambda\gamma^\nu\gamma^\rho\gamma^\sigma) \;+\; g^{\kappa\nu}\times\mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\rho\gamma^\sigma)\\ & -\; g^{\kappa\rho}\times\mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\sigma) \;+\; g^{\kappa\sigma}\times\mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho)\\ =\; & 4g^{\kappa\lambda}\times\left(g^{\mu\nu}g^{\rho\sigma} \;-\; g^{\mu\rho}g^{\nu\sigma} \;+\; g^{\mu\sigma}g^{\nu\rho}\right)\\ & -\; 4g^{\kappa\mu}\times\left(g^{\lambda\nu}g^{\rho\sigma} \;-\; g^{\lambda\rho}g^{\nu\sigma} \;+\; g^{\lambda\sigma}g^{\nu\rho}\right)\\ & +\; 4g^{\kappa\nu}\times\left(g^{\lambda\mu}g^{\rho\sigma} \;-\; g^{\lambda\rho}g^{\mu\sigma} \;+\; g^{\lambda\sigma}g^{\mu\rho}\right)\\ & -\; 4g^{\kappa\rho}\times\left(g^{\lambda\mu}g^{\nu\sigma} \;-\; g^{\lambda\nu}g^{\mu\sigma} \;+\; g^{\lambda\sigma}g^{\mu\nu}\right)\\ & +\; 4g^{\kappa\sigma}\times\left(g^{\lambda\mu}g^{\nu\rho} \;-\; g^{\lambda\nu}g^{\mu\rho} \;+\; g^{\lambda\rho}g^{\mu\nu}\right).\end{aligned} \tag{40}$$

For products of more γ matrices, the recursive formulae (39) for traces produce even more terms (105 terms for $n = 8$, 945 terms for $n = 10$, *etc.*, *etc.*), so it helps to reduce n whenever possible. For example, if the matrix product inside the trace contains two $\not{a}$ matrices (for the same 4–vector a^μ) next to each other, you can simplify the product using $\not{a}\,\not{a} = a^2$, thus

$$\operatorname{tr}(\not{a}\,\not{a}\,\not{b}\cdots\not{c}) \;=\; a^2 \times \operatorname{tr}(\not{b}\cdots\not{c}). \tag{41}$$

Also, when a product contains γ^α and γ_α with the same Lorentz index α which should be summed over, we may simplify the trace using

$$\gamma^\alpha\gamma_\alpha \;=\; 4, \quad \gamma^\alpha\not{a}\gamma_\alpha \;=\; -2\,\not{a}, \quad \gamma^\alpha\not{a}\,\not{b}\gamma_\alpha \;=\; +4(ab), \quad \gamma^\alpha\not{a}\,\not{b}\,\not{c}\gamma_\alpha \;=\; -2\,\not{c}\,\not{b}\,\not{a}, \tag{42}$$

etc., *cf.* homework set#7, problem 1(b).

In the electroweak theory, one often needs to calculate traces of products containing the γ^5 matrix. If the γ^5 appears more than once, we may simplify the product using $\gamma^5\gamma^5 = 1$ and $\gamma^5\gamma^\nu = -\gamma^\nu\gamma^5$, hence

$$(1 \pm \gamma^5)^2 \;=\; 1 \;\pm\; 2\gamma^5 \;+\; \gamma^5\gamma^5 \;=\; 2(1 \pm \gamma^5), \tag{43}$$

$$(1 \pm \gamma^5)(1 \mp \gamma^5) \;=\; 1 \;-\; \gamma^5\gamma^5 \;=\; 0, \tag{44}$$

$$\gamma^\nu(1 \pm \gamma^5) \;=\; (1 \mp \gamma^5)\gamma^\nu. \tag{45}$$

For example,

$$\begin{aligned}
\operatorname{tr}\Big(\gamma^\mu(1-\gamma^5)\,\not{p}\,\gamma^\nu(1-\gamma^5)\,\not{q}\Big) \;&=\; \operatorname{tr}\Big(\gamma^\mu(1-\gamma^5)\,\not{p}\,(1+\gamma^5)\gamma^\nu\,\not{q}\Big)\\
&=\; \operatorname{tr}\Big(\gamma^\mu(1-\gamma^5)(1-\gamma^5)\,\not{p}\,\gamma^\nu\,\not{q}\Big)\\
&=\; 2\operatorname{tr}\Big(\gamma^\mu(1-\gamma^5)\,\not{p}\,\gamma^\nu\,\not{q}\Big)\\
&=\; 2\operatorname{tr}\Big((1+\gamma^5)\gamma^\mu\,\not{p}\,\gamma^\nu\,\not{q}\Big)\\
&=\; 2\operatorname{tr}\Big(\gamma^\mu\,\not{p}\,\gamma^\nu\,\not{q}\Big) \;+\; 2\operatorname{tr}\Big(\gamma^5\gamma^\mu\,\not{p}\,\gamma^\nu\,\not{q}\Big).
\end{aligned} \tag{46}$$

When the γ^5 appears just one time, we may use $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ to show that

$$\operatorname{tr}(\gamma^5) \;=\; 0, \quad \operatorname{tr}(\gamma^5\gamma^\nu) \;=\; 0, \quad \operatorname{tr}(\gamma^5\gamma^\mu\gamma^\nu) \;=\; 0, \quad \operatorname{tr}(\gamma^5\gamma^\lambda\gamma^\mu\gamma^\nu) \;=\; 0, \tag{47}$$

while

$$\mathrm{tr}(\gamma^5\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu) \;=\; -4i\epsilon^{\kappa\lambda\mu\nu}. \tag{48}$$

For more γ^ν matrices accompanying the γ^5 we have

$$\mathrm{tr}(\gamma^5\gamma^{\nu_1}\cdots\gamma^{\nu_n}) \;=\; 0 \quad \forall\text{ odd } n. \tag{49}$$

Finally, for even $n = 6, 8, \ldots$ numbers of γ^ν's there are recursive formulae based on the identity

$$\gamma^5\gamma^\lambda\gamma^\mu\gamma^\nu \;=\; -g^{\lambda\mu}\times\gamma^5\gamma^\nu + \; g^{\lambda\nu}\times\gamma^5\gamma^\mu - \; g^{\mu\nu}\times\gamma^5\gamma^\lambda + \; i\epsilon^{\lambda\mu\nu\rho}\times\gamma_\rho\,. \tag{50}$$

For example, for $n = 6$

$$\begin{aligned}
\mathrm{tr}(\gamma^5\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\gamma) \;&=\; \mathrm{tr}(\gamma^5\gamma^\lambda\gamma^\mu\gamma^\nu\times\gamma^\alpha\gamma^\beta\gamma^\gamma)\\
&=\; -g^{\lambda\mu}\times\mathrm{tr}(\gamma^5\gamma^\nu\times\gamma^\alpha\gamma^\beta\gamma^\gamma) \;+\; g^{\lambda\nu}\times\mathrm{tr}(\gamma^5\gamma^\mu\times\gamma^\alpha\gamma^\beta\gamma^\gamma)\\
&\qquad -\; g^{\mu\nu}\times\mathrm{tr}(\gamma^5\gamma^\lambda\times\gamma^\alpha\gamma^\beta\gamma^\gamma) \;+\; i\epsilon^{\lambda\mu\nu\rho}\times\mathrm{tr}(\gamma_\rho\times\gamma^\alpha\gamma^\beta\gamma^\gamma)\\
&=\; -g^{\lambda\mu}\times(-4i)\epsilon^{\nu\alpha\beta\gamma} \;+\; g^{\lambda\nu}\times(-4i)\epsilon^{\mu\alpha\beta\gamma} \;-\; g^{\mu\nu}\times(-4i)\epsilon^{\lambda\alpha\beta\gamma}\\
&\qquad +\; i\epsilon^{\lambda\mu\nu\rho}\times 4\bigl(\delta^\alpha_\rho g^{\beta\gamma} \;-\; \delta^\beta_\rho g^{\alpha\gamma} \;+\; \delta^\gamma_\rho g^{\alpha\beta}\bigr)\\
&=\; 4i\Bigl(g^{\lambda\mu}\epsilon^{\nu\alpha\beta\gamma} \;-\; g^{\lambda\nu}\epsilon^{\mu\alpha\beta\gamma} \;+\; g^{\mu\nu}\epsilon^{\lambda\alpha\beta\gamma}\\
&\qquad\qquad +\; g^{\beta\gamma}\epsilon^{\lambda\mu\nu\alpha} \;-\; g^{\alpha\gamma}\epsilon^{\beta\lambda\mu\nu} \;+\; g^{\alpha\beta}\epsilon^{\gamma\lambda\mu\nu}\Bigr)
\end{aligned} \tag{51}$$

Back to Muon Pair Production

As an example of trace technology, let us calculate the traces

$$\overline{|\mathcal{M}|^2} \;=\; \frac{e^4}{4s^2}\times\mathrm{tr}\Bigl((\not p_1' + M_\mu)\,\gamma^\nu\,(\not p_2' - M_\mu)\,\gamma^\lambda\Bigr)\times\mathrm{tr}\Bigl((\not p_2 - m_e)\,\gamma_\nu\,(\not p_1 + m_e)\,\gamma_\lambda\Bigr). \tag{25}$$

for the muon pair production. Let's start with the trace due to summing over muons' spins,

$$\begin{aligned}
\mathrm{tr}\Bigl((\not p_1' + M_\mu)\gamma^\lambda(\not p_2' - M_\mu)\gamma^\nu\Bigr) \;&=\; \mathrm{tr}(\not p_1'\gamma^\lambda\not p_2'\gamma^\nu)\\
&\quad +\; M_\mu\times\mathrm{tr}(\gamma^\lambda\not p_2'\gamma^\nu) \;-\; M_\mu\times\mathrm{tr}(\not p_1'\gamma^\lambda\gamma^\nu)\\
&\quad -\; M_\mu^2\times\mathrm{tr}(\gamma^\lambda\gamma^\nu).
\end{aligned} \tag{52}$$

On the second line here, we have three γ matrices inside each trace, so those traces vanish. On the third line, $\mathrm{tr}(\gamma^\lambda\gamma^\nu) = 4g^{\lambda\nu}$. Finally, the trace on the first line follows from eq. (37),

$$\begin{aligned}
\mathrm{tr}(\not{p}'_1\gamma^\lambda\not{p}'_2\gamma^\nu) &= p'_{1\alpha}p'_{2\beta}\times\mathrm{tr}(\gamma^\alpha\gamma^\lambda\gamma^\beta\gamma^\nu) \\
&= p'_{1\alpha}p'_{2\beta}\times 4\Big(g^{\alpha\lambda}\times g^{\beta\nu} - g^{\alpha\beta}\times g^{\lambda\nu} + g^{\alpha\nu}\times g^{\lambda\beta}\Big) \\
&= 4p_1'^\lambda\times p_2'^\nu - 4(p'_1p'_2)\times g^{\lambda\nu} + 4p_1'^\nu\times p_2'^\lambda\,.
\end{aligned} \tag{53}$$

Altogether,

$$\begin{aligned}
\mathrm{tr}\Big((\not{p}'_1+M_\mu)\gamma^\lambda(\not{p}'_2-M_\mu)\gamma^\nu\Big) &= 4p_1'^\lambda p_2'^\nu + 4p_1'^\nu p_2'^\lambda - 4(p'_1p'_2)g^{\lambda\nu} - 4M_\mu^2 g^{\lambda\nu} \\
&= 4p_1'^\lambda p_2'^\nu + 4p_1'^\nu p_2'^\lambda - 2s\times g^{\lambda\nu}
\end{aligned} \tag{54}$$

where the last line follows from

$$s \equiv (p'_1+p'_2)^2 = p_1'^2 + p_2'^2 + 2(p'_1p'_2) = 2M_\mu^2 + 2(p'_1p'_2). \tag{55}$$

Similarly, for the second trace (25) due to averaging over electron's and positron's spins, we have

$$\begin{aligned}
\mathrm{tr}\Big((\not{p}_2-m_e)\,\gamma_\nu\,(\not{p}_1+m_e)\,\gamma_\lambda\Big) &= \mathrm{tr}(\not{p}_2\gamma_\nu\not{p}_1\gamma_\lambda) \\
&\quad + m_e\times\mathrm{tr}(\not{p}_2\gamma_\nu\gamma_\lambda) - m_e\times\mathrm{tr}(\gamma_\nu\not{p}_1\gamma_\lambda) \\
&\quad - m_e^2\times\mathrm{tr}(\gamma_\nu\gamma_\lambda) \\
&= 4p_{2\nu}\times p_{1\lambda} - 4(p_2p_1)\times g_{\nu\lambda} + 4p_{2\lambda}\times p_{1\nu} \\
&\quad + m_e\times 0 - m_e\times 0 - m_e^2\times 4g_{\nu\lambda} \\
&= 4p_{2\nu}p_{1\lambda} + 4p_{2\lambda}p_{1\nu} - 4((p_2p_1)+m_e^2)\times g_{\lambda\nu} \\
&= 4p_{2\nu}p_{1\lambda} + 4p_{2\lambda}p_{1\nu} - 2s\times g_{\lambda\nu}\,,
\end{aligned} \tag{56}$$

where the last line follows from

$$s \equiv (p_1+p_2)^2 = p_1^2 + p_2^2 + 2(p_2p_1) = 2m_e^2 + 2(p_2p_1). \tag{57}$$

It remains to multiply the two traces and sum over the Lorentz indices λ and ν:

$$\begin{aligned}
\operatorname{tr}\Big((\not p_1' + M_\mu)\gamma^\nu(\not p_2' - M_\mu)\gamma^\lambda\Big) \;\times \operatorname{tr}\Big((\not p_2 - m_e)\,\gamma_\nu\,(\not p_1 + m_e)\,\gamma_\lambda\Big) \;=& \\
=\;& \Big(4p_1'^\nu p_2'^\lambda \,+\, 4p_1'^\lambda p_2'^\nu \,-\, 2s\times g^{\nu\lambda}\Big)\times\Big(4p_{2\nu}p_{1\nu} \,+\, 4p_{2\lambda}p_{1\nu} \,-\, 2s\times g_{\nu\lambda}\Big) \\
=\;& 16(p_1'^\nu p_2'^\lambda \,+\, p_1'^\lambda p_2'^\nu)\times(p_{2\nu}p_{1\lambda} \,+\, p_{2\lambda}p_{1\nu}) \\
& -8s\,g^{\nu\lambda}\times(p_{2\nu}p_{1\lambda} \,+\, p_{2\lambda}p_{1\nu}) \\
& -8s\,g_{\nu\lambda}\times(p_1'^\lambda p_2'^\nu \,+\, p_1'^\nu p_2'^\lambda) \\
& +4s^2\times g^{\lambda\nu}g_{\lambda\nu} \\
=\;& 16\times 2\times\Big((p_1'p_1)(p_2'p_2) \,+\, (p_2'p_1)(p_1'p_2)\Big) \\
& -\; 8s\times 2(p_1p_2) \;-\; 8s\times 2(p_1'p_2') \;+\; 4s^2\times 4 \\
=\;& 32(p_1'p_1)(p_2'p_2) \;+\; 32(p_2'p_1)(p_1'p_2) \;+\; 16s\times(M_\mu^2+m_e^2)
\end{aligned} \tag{58}$$

The last line here follows from eqs. (55) and (57); indeed,

$$\begin{aligned}
2s \;-\; 2(p_1p_2) \;-\; 2(p_1'p_2') \;&=\; 2s \;-\; (s-2m_e^2) \;-\; 2(s-2M_\mu^2) \\
&=\; 2(M_\mu^2+m_e^2)
\end{aligned} \tag{59}$$

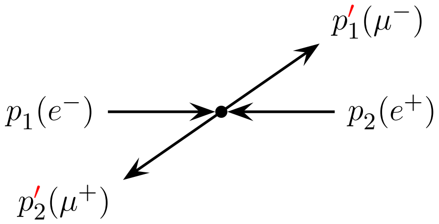
$$\Downarrow$$

$$4s^2\times 4 \;-\; 8s\times 2(p_1p_2) \;-\; 8s\times 2(p_1'p_2') \;=\; 16s\times(M_\mu^2+m_e^2). \tag{60}$$

Finally, plugging eq. (58) into eq. (25), we find that for muon pair production in electron-positron collisions

$$\overline{|\mathcal{M}|^2} \;\equiv\; \frac{1}{4}\sum_{\text{all spins}}|\mathcal{M}|^2 \;=\; \frac{4e^4}{s^2}\times\Big(2(p_1'p_1)(p_2'p_2) \,+\, 2(p_2'p_1)(p_1'p_2) \,+\, s(M_\mu^2+m_e^2)\Big). \tag{61}$$

At this point, we are done with the spins and the traceology, and all that's left to work out is the kinematics of pair production. In most electron-positron colliders, the electron and the positron beams have equal energies and the particles collide head-on, so the detectors see the collision in the center of mass frame. In this frame



$$p^{\mu}_{1,2} \;=\; (E, \pm\mathbf{p}), \quad p'^{\mu}_{1,2} \;=\; (E', \pm\mathbf{p}')$$
$$\text{for } E' \;=\; E \;=\; \tfrac{1}{2}E_{\rm cm} \quad \text{but } \mathbf{p}' \;\neq\; \mathbf{p}. \tag{62}$$

Consequently,

$$\begin{aligned}
(p_1'p_1) \;=\; (p_2'p_2) \;&=\; E^2 \;-\; \mathbf{p}'\cdot\mathbf{p}, \\
(p_2'p_1) \;=\; (p_1'p_2) \;&=\; E^2 \;+\; \mathbf{p}'\cdot\mathbf{p}, \\
s \;&=\; 4E^2, \\
2(p_1'p_1)(p_2'p_2) \;+\; 2(p_2'p_1)(p_1'p_2) \;&=\; 2(E^2-\mathbf{p}'\cdot\mathbf{p})^2 \;+\; 2(E^2+\mathbf{p}'\cdot\mathbf{p})^2 \\
&=\; 4E^4 \;+\; 4(\mathbf{p}'\cdot\mathbf{p})^2 \\
&=\; 4E^4 \;+\; 4\mathbf{p}'^2\,\mathbf{p}^2\times\cos^2\theta,
\end{aligned} \tag{63}$$

and therefore

$$\overline{|\mathcal{M}|^2} \;=\; e^4\left(1 \;+\; \frac{\mathbf{p}'^2\,\mathbf{p}^2}{E^4}\times\cos^2\theta \;+\; \frac{M_\mu^2+m_e^2}{E^2}\right). \tag{64}$$

where $\mathbf{p}'^2 = E^2 - M_\mu^2$ and $\mathbf{p}^2 = E^2 - m_e^2$.

We may simplify this expression a bit using the experimental fact that the muon is much heavier than the electron, $M_\mu \approx 207m_e$, so we need ultra-relativistic $e^\mp$ to produce $\mu^\mp$, $E > M_\mu \gg m_e$. This allows us to neglect the m_e^2 term in eq. (64) and let $\mathbf{p}^2 = E^2$, thus

$$\overline{|\mathcal{M}|^2} \;=\; e^4\left(\left(1 \;+\; \frac{M_\mu^2}{E^2}\right) \;+\; \left(1 \;-\; \frac{M_\mu^2}{E^2}\right)\times\cos^2\theta\right), \tag{65}$$

and consequently the partial cross-section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm c.m.} \;=\; \frac{\alpha^2}{4s}\times\left(\left(1 \;+\; \frac{M_\mu^2}{E^2}\right) \;+\; \left(1 \;-\; \frac{M_\mu^2}{E^2}\right)\times\cos^2\theta\right)\times\sqrt{1 \;-\; \frac{M_\mu^2}{E^2}} \tag{66}$$

where the root comes from the phase-space factor $|\mathbf{p}'|/|\mathbf{p}|$ for inelastic processes.

Looking at the angular dependence of this partial cross-section, we see that just above the energy threshold, for $E = M_\mu +$small, the muons are produced isotropically in all directions. On the other hand, for very high energies $E \gg M_\mu$ when all 4 particles are ultra-relativistic,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} \propto\ 1\,+\,\cos^2\theta. \tag{67}$$

In the current homework set#10 (problem 2) you will see that the polarized cross sections depend on the angle as $(1\pm\cos\theta)^2$ where the sign $\pm$ depends on the helicities of initial and final particles; for the un-polarized particles, we average / sum over helicities, and that produces the averaged angular distribution $1+\cos^2\theta$.

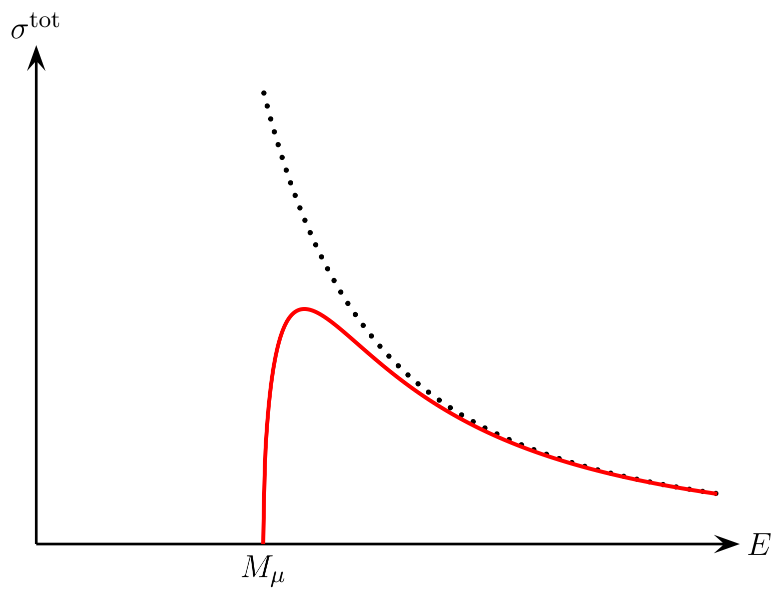
Finally, the total cross-section of muon pair production follows from eq. (66) using

$$\int\! d^2\Omega\ =\ 4\pi, \qquad \int\! d^2\Omega\,\cos^2\theta\ =\ \frac{4\pi}{3}\,, \tag{68}$$

hence

$$\sigma^{\text{tot}}(e^-e^+\to\mu^-\mu^+)\ =\ \frac{4\pi}{3}\,\frac{\alpha^2}{s}\times\left(1\,+\,\frac{M_\mu^2}{2E^2}\right)\sqrt{1\,-\,\frac{M_\mu^2}{E^2}}\,. \tag{69}$$

At the threshold this cross-section is zero, but it rises very rapidly with energy and reaches the maximum value at $E\approx 1.073\ M_\mu$; after that, it starts decreasing due to the overall $1/s$ factor. Here is the plot:



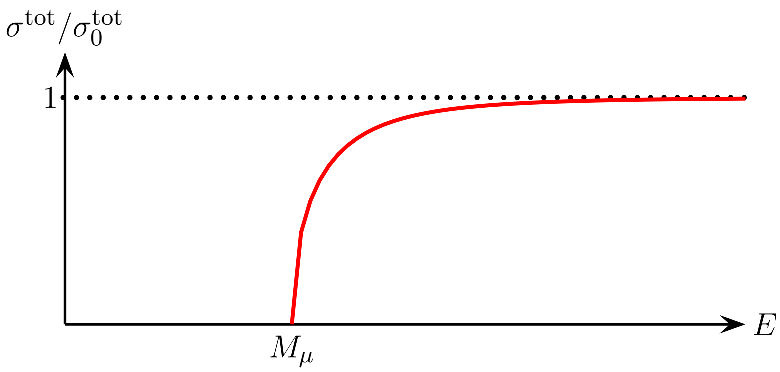
The red line here shows the actual total cross-section while the dotted black line shows the approximation

$$\sigma^{\rm tot} \approx \sigma_0^{\rm tot} \equiv \frac{4\pi}{3}\frac{\alpha^2}{s} \tag{70}$$

for energies well above the threshold. This approximate formula rapidly approaches the actual cross-section (69),

$$\frac{\sigma_{\rm actual}^{\rm total}}{\sigma_0^{\rm total}} = 1 - O\left(\left(\frac{M_\mu}{E}\right)^4\right), \tag{71}$$

so even for energy E just 50% above the threshold the approximation (70) overestimates the actual cross-section by only 10%. Here is the plot of the $\sigma_{\rm actual}^{\rm tot}/\sigma_0^{\rm tot}$ as a function of the electron energy:



Quark Pair Production

High energy electron-positron collisions can pair-produce all kinds of charged quanta: the muons $\mu^- + \mu^+$, the tau leptons $\tau^- + \tau^+$, and the quark-antiquark pairs $q + \bar{q}$. The tau pair production works exactly similar to the muon pair production, except for the higher energy threshold due to much larger mass $M_\tau \approx 1.78$ GeV of the tau lepton. The production or quark-antiquark pairs is also similar to the muon pair production, but there are some differences due to:

1. Quarks coming in 3 colors and several flavors and having different electric charges than the muon or tau.

2. Strong interactions between the quark and the antiquark.

The first issue is easy to deal with. Since the quark-photon vertex is proportional to the quark's charge Q, the amplitude becomes

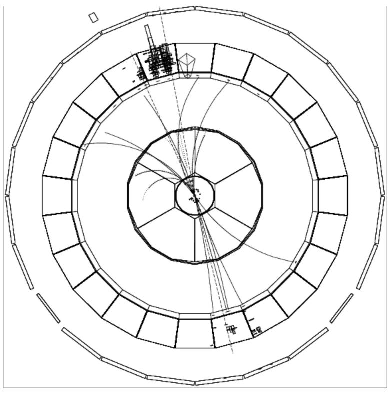
$$\langle q,\bar{q}|\,\mathcal{M}\,|e^-,e^+\rangle \;=\; \frac{-eQ}{s}\times\bar{u}(q)\gamma^\nu v(\bar{q})\times\bar{v}(e^+)\gamma_\nu u(e^-) \tag{72}$$

which is different from the muon production amplitude by a factor $-Q/e$. Also, the cross-section needs to be summed over the quark colors and flavors, but the sum over flavors should be limited to the quarks of masses $m < \frac{1}{2}E_{\rm cm}$. Thus, at the tree level and at energies far enough from the quark mass thresholds,

$$\sigma_{\rm tree}(e^- + e^+ \to q + \bar{q}) \;=\; \sigma_0(E_{\rm cm})\times\sum_{\substack{\text{flavors with}\\ \text{mass}<\frac{1}{2}E_{\rm cm}}}(-Q/e)^2\times 3(\text{colors}) \tag{73}$$

where $\sigma_0(E_{\rm cm})$ is the total cross-section (70) for the muon pair production.

The strong interactions have several effects, especially the *confinement*: the quarks (and the antiquarks) can never be free. When the quark and the antiquark produced in the e^-+e^+ collision start flying away from each other, they create a flux tube of the chromo-electric field which pulls them back towards each other with a 16-ton force. However, this flux tube is unstable, and when it breaks it creates creates several additional quark-antiquark pairs; eventually all these quarks and antiquarks turn into a bunch of hadrons (mesons or baryons). At high energies $E_{\rm cm} \gtrsim 10$ GeV, these hadrons form two *jets* in the directions of the original quark and antiquark. Here is a picture of such a 2-jet event from the ALEPH detector at LEP at CERN:



AT lower energies $E_{\rm cm} \lesssim 10$ GeV, the jets become too fat to identify as jets, and all we see is a bunch of hadrons flying away in all directions from the collision point.

However, when we measure the total cross-section for producing all kinds of hadrons in electron-positron collisions, it turns out to be pretty much similar to the total cross-section for making $q+\bar{q}$ pairs,

$$\sigma^{\text{total}}(e^- + e^+ \to \text{hadrons}) \;\approx\; \sigma_{\text{total}}(e^- + e^+ \to q + \bar{q}). \tag{74}$$

To factor out the $1/E_{\text{cm}}^2$ energy dependence and other QED factors, the experimentalists often report the R ration of this total cross-section to the cross-section of muon pair production,

$$R \;=\; \frac{\sigma^{\text{tot}}(e^- + e^+ \to \text{hadrons})}{\sigma_0^{\text{tot}}(e^- + e^+ \to \mu^- + \mu^+)} \;\approx\; \frac{\sigma^{\text{tot}}(e^- + e^+ \to q + \bar{q})}{\sigma_0^{\text{tot}}(e^- + e^+ \to \mu^- + \mu^+)}\,. \tag{75}$$

In light of eq. (73), at the tree level and at energies far enough from the quark mass thresholds,

$$R_{\text{tree}}(E_{\text{cm}}) \;=\; \sum_{\substack{\text{flavors with}\\ \text{mass}<\frac{1}{2}E_{\text{cm}}}} (-Q/e)^2 \times 3. \tag{76}$$

- At energies $E_{\text{cm}} < 2m_c \approx 3$ GeV, only 3 quark flavors are at play, namely 'up' u, 'down' d, and 'strange' s, whose electric charges are respectively $+\frac{2}{3}e$, $-\frac{1}{3}e$, and $-\frac{1}{3}e$. Consequently,

$$R_{\text{tree}} \;=\; \Big((\tfrac{2}{3})^2 + (\tfrac{1}{3})^2 + (\tfrac{1}{3})^2\Big) \times 3 \;=\; 2. \tag{77}$$

- At energies above $2m_c \approx 3$ GeV but below $2m_b \approx 9$ GeV, we add a fourth quark quark flavor — the 'charm' c of charge $+\frac{2}{3}e$. Consequently, in this energy range

$$R_{\text{tree}} \;=\; \Big((\tfrac{2}{3})^2 + (\tfrac{1}{3})^2 + (\tfrac{1}{3})^2 + (\tfrac{2}{3})^2\Big) \times 3 \;=\; \frac{10}{3}\,. \tag{78}$$

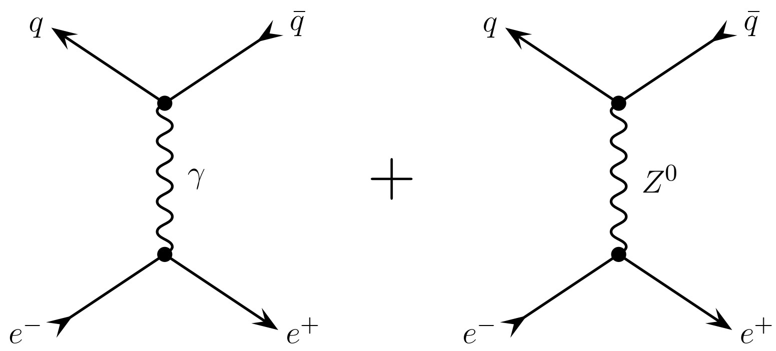
- Finally, at energies above $2m_b \approx 9$ GeV, we add the fifth flavor — 'bottom' or 'beauty' b of charge $-\frac{1}{3}e$. Consequently, the R ration becomes

$$R_{\text{tree}} \;=\; \Big((\tfrac{2}{3})^2 + (\tfrac{1}{3})^2 + (\tfrac{1}{3})^2 + (\tfrac{2}{3})^2 + (\tfrac{1}{3})\Big) \times 3 \;=\; \frac{11}{3}\,. \tag{79}$$

⊗ In principle, we should continue this game to the next quark threshold $2m_t \approx 346$ GeV where the sixth quark flavor — 'top' or 'truth' t, charge $+\frac{2}{3}e$ — becomes available.

However, at much lower energies $E_{\text{cm}} \sim 40$ GeV we start getting interference from pair production involving a virtual Z^0 vector boson instead of the virtual photon, thus even at the tree level

$$i\mathcal{M} \;=\; \qquad \text{(80)}$$



and hence a much more complicated formula for the R ratio.

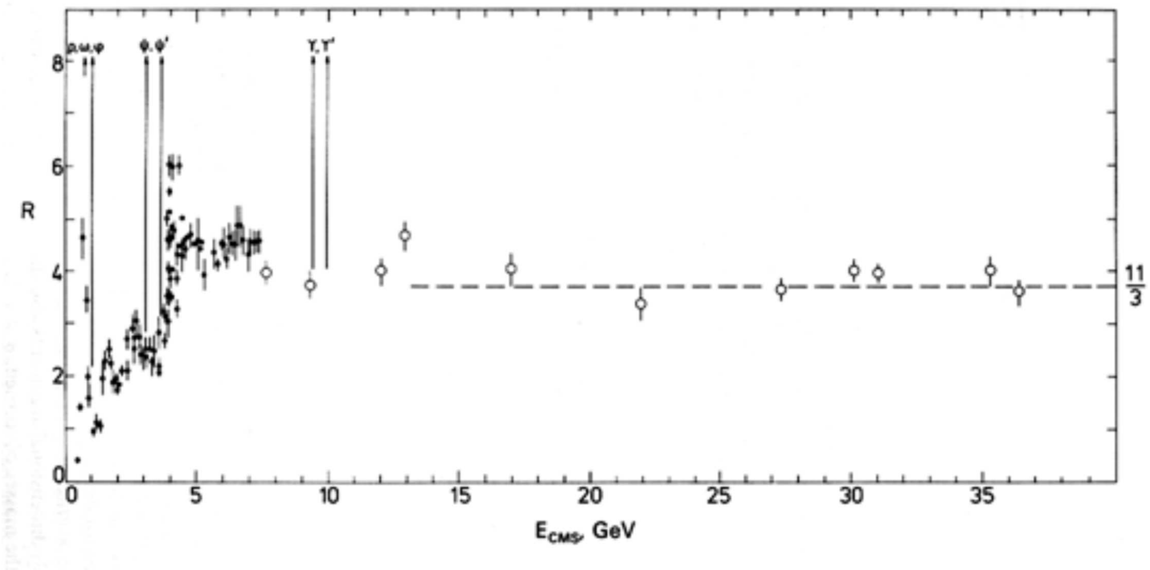
Taking the strong interactions into account has two effects on the R ratio:

⋆ Perturbatively, there are corrections due to loops involving virtual gluons or emission of real gluons (which can form extra jets). We are not going to calculate such corrections in this class, so let me simply summarize their net effect:

$$R(E_{\text{cm}}) \;=\; R_{\text{tree}}(E_{\text{cm}}) \times \left(1 + \frac{\alpha_{\text{QCD}}(E_{\text{cm}})}{\pi} + O(\alpha_{\text{QCD}}^2)\right), \tag{81}$$

where $\alpha_{\text{QCD}} = g^2[SU(3)]/4\pi$ is the QCD coupling, or rather the renormalized QCD coupling which is rather strong at low energies $E \lesssim 1$ GeV but gets weaker at higher energies; numerically $\alpha_{\text{QCD}}(3\text{ GeV}) \approx 0.32$ while $\alpha_{\text{QCD}}(40\text{ GeV}) \approx 0.16$, so the corrections in eq. (80) increase R by a few percent above its tree-level value.

⋆ Non-perturbatively, there are sharp resonances due to $q\bar{q}$ bound states such as J/ψ at 3.097 GeV or Y at 9.460 GeV, and when you tune the center-of-mass energy of the electron-positron collision to the muss of such a resonance, you get a *much* higher cross-section for making hadrons. In terms of the experimental R ratio, it can suddenly jump to a few thousands when you hit a resonance and then drop back to its usual value from eqs. (79) and (81).



Here (or perhaps at the top of next page) is the experimental plot of the R ration as a function of the center-of-mass energy:

Note sharp resonances near 1 GeV due to ϕ (an $S\bar{s}$ vector meson), between 3 and 4 GeV due to J/ψ, ψ', and ψ'' charmonium states, and between 9 and 10 GeV due to $b\bar{b}$ due to Y, Y', *etc.* bound states. Away from the resonances, we have $R \approx 2$ between 1 and 3 GeV, $R \approx \frac{10}{3}$ between 4 and 9 GeV, and $R \approx \frac{11}{3}$ above 10 GeV, in good agreement with eq. (79).