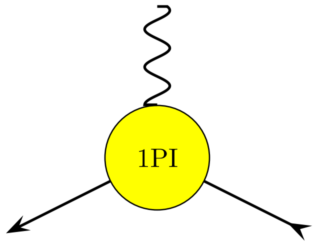
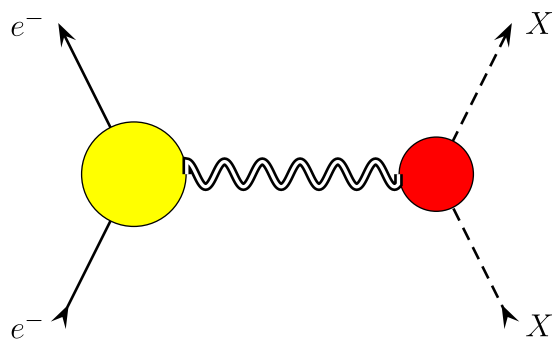
QED Vertex Correction

Consider the dressed electron-electron-photon vertex in QED,

$$ie\Gamma^\mu(p',p) \;=\;$$  (1)

We are interested in this vertex in the context of elastic Coulomb scattering,



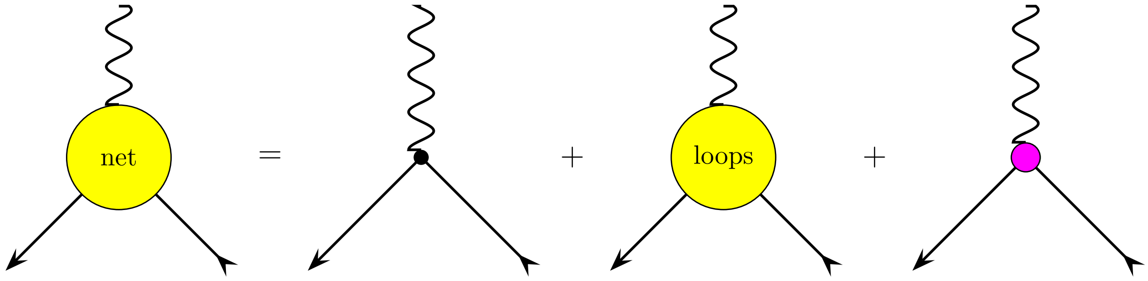
(2)

so we take the incoming and the outgoing electrons to be on-shell, $p^2 = p'^2 = m^2$, but the photon is off-shell, $q^2 \neq 0$. Moreover, we put the vertex in the context of the complete electron line — including the external line factors, thus $\bar{u}(p') \times ie\Gamma^\mu \times u(p)$. As we saw in the my notes on the form factors, this simplifies the Lorentz and Dirac structure of the vertex and allows us to write it as

$$\Gamma^\mu(p',p) \;=\; F_{\rm el}(q^2) \times \frac{(p'+p)^\mu}{2m} \;+\; F_{\rm mag}(q^2) \times \frac{i\sigma^{\mu\nu}q_\nu}{2m} \;=\; F_1(q^2) \times \gamma^\mu \;+\; F_2(q^2) \times \frac{i\sigma^{\mu\nu}q_\nu}{2m}\,. \quad (3)$$

At the tree level, the electron is a point-like spin $= \frac{1}{2}$ particle obeying Dirac equations, hence $F_1(q^2) \equiv 1$ and $F_2(q^2) \equiv 0$. But the quantum corrections in QED mix the elementary electron state with the composite states like $|e^-\gamma\rangle$, $|e^-e^-e^+\rangle$, *etc.*, and this leads to the non-trivial q^2–dependent form-factors.

In these notes we shall calculate the $F_1(q^2)$ and the $F_2(q^2)$ form-factors to the one-loop order in QED. But before we draw and evaluate specific loop diagrams, note that in the counterterm perturbation theory



(4)

thus

$$ie\Gamma^\mu_{\rm net}(p',p) \;=\; (ie\gamma^\mu)_{\rm tree} + \; ie\Gamma^\mu_{\rm loops}(p',p) \;+\; ie\delta_1\times\gamma^\mu, \tag{5}$$

where the counterterm coefficient δ_1 is set such that

$$\Gamma^\mu_{\rm net}(\text{on-shell } p'=p) \;=\; \gamma^\mu,\ \text{exactly}. \tag{6}$$

For on-shell incoming and outgoing electrons but off-shell photon, eq. (5) translates to the counterterm language as

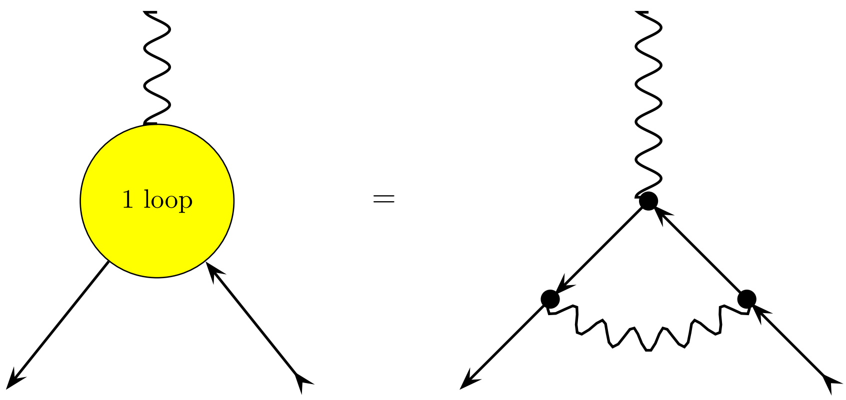
$$\begin{aligned} F_1^{\rm net}(q^2) &= 1^{\rm tree} \;+\; F_1^{\rm loops}(q^2) \;+\; \delta_1\,, \\ F_2^{\rm net}(q^2) &= 0^{\rm tree} \;+\; F_2^{\rm loops}(q^2), \end{aligned} \tag{7}$$

while eq. (6) becomes

$$\delta_1 \;=\; -F_1^{\rm loops}(q^2=0) \quad\Longrightarrow\quad F_1^{\rm net}(q^2=0) \;=\; 1. \tag{8}$$

One-Loop Calculation: Working Through the Algebra

Now let's turn to the actual calculation of the dressed QED vertex $\Gamma^\mu_{\rm net}$ to the order $O(\alpha)$. Fortunately for us, at this order there is only one 1-loop diagram dressing up the vertex, namely



(9)

In the Feynman gauge for the internal photon's propagator, this diagram evaluates to

$$\begin{aligned}
ie\Gamma^\mu_{1\,\text{loop}}(p',p) &= \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\nu\lambda}}{k^2+i0} \times ie\gamma_\nu \times \frac{i}{\not p' + \not k - m + i0} \times ie\gamma^\mu \times \frac{i}{\not p + \not k - m + i0} \times ie\gamma_\lambda \\
&= e^3\int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+i0} \times \gamma^\nu \times \frac{\not p' + \not k + m}{(p'+k)^2 - m^2 + i0} \times \gamma^\mu \times \frac{\not p + \not k + m}{(p+k)^2 - m^2 + i0} \times \gamma_\nu \\
&= e^3\int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{\mathcal{D}}
\end{aligned} \tag{10}$$

where

$$\mathcal{N}^\mu = \gamma^\nu(\not k + \not p' + m)\gamma^\mu(\not k + \not p + m)\gamma_\nu \tag{11}$$

and

$$\mathcal{D} = \left[k^2 + i0\right] \times \left[(p+k)^2 - m^2 + i0\right] \times \left[(p'+k)^2 - m^2 + i0\right]. \tag{12}$$

The purpose of this section of the notes is to simplify these numerator and denominator. Using the Feynman parameter trick, we may combine the 3 denominator factors as

$$\frac{1}{\mathcal{D}} = \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{\left[x((p+k)^2-m^2) \,+\, y((p'+k)^2-m^2) \,+\, z(k^2) \,+\, i0\right]^3}. \tag{13}$$

Inside the big square brackets here we have

$$\begin{aligned}
[\cdots] &= x\times\left((p+k)^2 - m^2\right) \,+\, y\times\left((p'+k)^2 - m^2\right) \,+\, z\times k^2 \\
&= k^2\times(x+y+z=1) \,+\, 2k_\mu(xp+yp')^\mu \,+\, x(p^2-m^2) \,+\, y(p'^2 - m^2) \\
&= (k+xp+yp')^2 \,-\, \Delta
\end{aligned} \tag{14}$$

where

$$\Delta = (xp+yp')^2 \,-\, xp^2 \,-\, yp'^2 \,+\, (x+y)m^2. \tag{15}$$

In this formula

$$\begin{aligned}
(xp+yp')^2 &= x^2p^2 \,+\, y^2p'^2 \,+\, xy\times\left(2p\cdot p' = p^2 + p'^2 - (p'-p)^2\right) \\
&= x(x+y)p^2 \,+\, y(y+x)p'^2 \,-\, xyq^2,
\end{aligned} \tag{16}$$

hence

$$\begin{aligned}\Delta &= x(x+y-1)p^2 \;+\; y(x+y-1)p'^2 \;-\; xyq^2 \;+\; (x+y)m^2 \\ &= -xzp^2 \;-\; yzp'^2 \;-\; xyq^2 \;+\; (1-z)m^2.\end{aligned} \tag{17}$$

Moreover, for the on-shell electron momenta, $p^2 = p'^2 = m^2$, we may further simplify

$$(1-z)\times m^2 \;-\; xz\times p^2 \;-\; yz\times p'^2 \;=\; m^2\times\Big((1-z) \;-\; (x+y=1-z)z \;=\; (1-z)^2\Big), \tag{18}$$

hence

$$\Delta \;=\; (1-z)^2\times m^2 \;-\; xy\times q^2. \tag{19}$$

Let us also define the shifted loop momentum

$$\ell \;=\; k \,+\, xp \,+\, yp', \tag{20}$$

then we can rewrite the denominator as

$$\frac{1}{\mathcal{D}} \;=\; \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{[\ell^2-\Delta+i0]^3}\,. \tag{21}$$

As usual, we plug this denominator into the loop integral (10), then change the order of integration — $\int$ over the loop momentum before $\int$ over the Feynman parameters, — and then shift the momentum integration variable from k to ℓ, thus

$$\Gamma^{\mu}_{1\,\text{loop}}(p',p) \;=\; -2ie^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{\big[\ell^2-\Delta+i0\big]^3}\,. \tag{22}$$

But to make full use of the momentum shift, we need to re-express the numerator $\mathcal{N}^\mu$ in terms of the shifted momentum ℓ. It would also help to simplify the numerator (11) *in the context of this monstrous integral.*

The first step towards simplifying the $\mathcal{N}^\mu$ is obvious: Let us get rid of the γ^ν and γ_ν factors using the γ matrix algebra, *eg.*, $\gamma^\nu \not{a} \gamma_\nu = -2\not{a}$, *etc.*. However, in order to allow for the dimensional regularization, we need to re-work the algebra for an arbitrary spacetime dimension D where $\gamma^\nu\gamma_\nu = D \neq 4$. So let X be some product of n Dirac matrices, then

$$\gamma^\nu X \gamma_\nu \;=\; [\gamma^\nu, X\}\gamma_\nu \;+\; (-1)^n \times X \times \bigl(\gamma^\nu\gamma_\nu = D\bigr), \tag{23}$$

where the (anti)commutator term is D independent and only the second term is affected by $D \neq 4$. Hence,

$$\Bigl(\gamma^\nu X \gamma_\nu\Bigr)_{D\neq 4} \;=\; \Bigl(\gamma^\nu X \gamma_\nu\Bigr)_{4d} \;+\; (-1)^n (D-4) \times X; \tag{24}$$

in particular, for $n = 1, 2, 3$ we have

$$\begin{aligned} \gamma^\nu \not{a} \gamma_\nu \;&=\; -2\not{a} \;+\; (4-D)\times \not{a}, \\ \gamma^\nu \not{a}\not{b} \gamma_\nu \;&=\; 4(ab) \;-\; (4-D)\times \not{a}\not{b}, \\ \gamma^\nu \not{a}\not{b}\not{c} \gamma_\nu \;&=\; -2\not{c}\not{b}\not{a} \;+\; (4-D)\times \not{a}\not{b}\not{c}. \end{aligned} \tag{25*}$$

Thus, applying these formulae to the numerator (11) we arrive at

$$\begin{aligned} \mathcal{N}^\mu \;\stackrel{\text{def}}{=}\; &= \; \gamma^\nu(\not{k} + \not{p}' + m)\gamma^\mu(\not{k} + \not{p} + m)\gamma_\nu \\ &= \; -2m^2\gamma^\mu \;+\; 4m(p' + p + 2k)^\mu \;-\; 2(\not{p} + \not{k})\gamma^\mu(\not{p}' + \not{k}) \\ &\quad +\; (4-D)(\not{p}' + \not{k} - m)\gamma^\mu(\not{p} + \not{k} - m). \end{aligned} \tag{26}$$

The second step is to re-express this numerator in terms of the loop momentum ℓ rather than k using eq. (20). Expanding the result in powers of ℓ, we get quadratic, linear and ℓ–independent terms, but the linear terms do not contribute to the $\int d^D\ell$ integral because they are odd with respect to $\ell \to -\ell$ while everything else in that integral is even. Consequently, *in the context of* *eq.* (22) we may neglect the linear terms, thus

$$\begin{aligned}
\mathcal{N}^\mu &= -2m^2\gamma^\mu + 4m(p' + p + 2\ell - 2xp - 2yp')^\mu \\
&\quad - 2(\not{p} + \not{\ell} - x\not{p} - y\not{p}')\gamma^\mu(\not{p}' + \not{\ell} - x\not{p} - y\not{p}') \\
&\quad + (4-D)(\not{p}' + \not{\ell} - x\not{p} - y\not{p}' - m)\gamma^\mu(\not{p} + \not{\ell} - x\not{p} - y\not{p}' - m) \\
&\quad \langle\!\langle \text{skipping terms linear in } \ell \rangle\!\rangle \\
&\cong -2m^2\gamma^\mu + 4m(p + p' - 2xp - 2yp')^\mu \\
&\quad - 2\not{\ell}\gamma^\mu\not{\ell} - 2(\not{p} - x\not{p} - y\not{p}')\gamma^\mu(\not{p}' - x\not{p} - y\not{p}') \\
&\quad + (4-D)\not{\ell}\gamma^\mu\not{\ell} + (4-D)(\not{p}' - y\not{p}' - x\not{p} - m)\gamma^\mu(\not{p} - x\not{p} - y\not{p}' - m).
\end{aligned} \tag{27}$$

Next, we make use of $p' - p = q$ and $1 - x - y = z$ to rewrite

$$\begin{aligned}
2xp + 2yp' &= (x+y)\times(p+p') + (x-y)\times(p-p'), \\
p + p' - 2xp - 2yp' &= z\times(p'+p) + (x-y)\times q, \\
p - xp - yp' &= z\times p - y\times q \\
&= z\times p' - (1-x)\times q, \\
p' - xp - yp' &= z\times p' + x\times q \\
&= z\times p + (1-y)\times q,
\end{aligned} \tag{28}$$

and consequently

$$\begin{aligned}
\mathcal{N}^\mu &\cong -2m^2\gamma^\mu + 4mz(p'+p)^\mu + 4m(x-y)q^\mu \\
&\quad + (-2+4-D)\times\not{\ell}\gamma^\mu\not{\ell} \\
&\quad - 2(z\not{p}' - (1-x)\not{q})\gamma^\mu(z\not{p} + (1-y)\not{q}) \\
&\quad + (4-D)(z\not{p}' + x\not{q} - m)\gamma^\mu(z\not{p} - y\not{q} - m).
\end{aligned} \tag{29}$$

The third step is to make use of the external fermions being on-shell. This means more than just $p^2 = p'^2 = m^2$: We also sandwich the vertex $ie\Gamma^\mu$ between the Dirac spinors $\bar{u}(p')$ on the left and $u(p)$ on the right. The two spinors satisfy the appropriate Dirac equations $\not{p}u(p) = mu(p)$ and $\bar{u}(p')\not{p}' = \bar{u}(p')m$, so in the context of $\bar{u}(p')\Gamma^\mu u(p)$,

$$A\times\not{p} \cong A\times m \quad \text{and} \quad \not{p}'\times B \cong m\times B \tag{30}$$

for any terms in Γ^μ that look like $A\times\not{p}$ or $\not{p}'\times B$ for some A or B. Consequently, the terms on the last two lines of eq. (29) are equivalent to

$$\begin{aligned}
(z\not{p}' - (1-x)\not{q})\,\gamma^\mu\,(z\not{p} + (1-y)\not{q}) \;&\cong\; (zm - (1-x)\not{q})\,\gamma^\mu\,(zm + (1-y)\not{q}) \\
&= z^2m^2\times\gamma^\mu \;-\; (1-x)(1-y)\times\not{q}\gamma^\mu\not{q} \\
&\quad +\; z(x-y)m\times\Big(\tfrac12\{\gamma^\mu,\not{q}\} \;=\; q^\mu\Big) \\
&\quad +\; z(2-x-y)m\times\Big(\tfrac12[\gamma^\mu,\not{q}] \;=\; -i\sigma^{\mu\nu}q_\nu\Big) \qquad (31)
\end{aligned}$$

where
$$z(2-x-y) \;=\; z(1+z), \qquad (32)$$
$$(1-x)(1-y) \;=\; 1\;-\;x\;-\;y\;+\;xy \;=\; z\;+\;xy, \qquad (33)$$

and
$$\begin{aligned}
(z\not{p}' + x\not{q} - m)\,\gamma^\mu\,(z\not{p} - y\not{q} - m) \;&\cong\; ((z-1)m + x\not{q})\,\gamma^\mu\,((z-1)m - y\not{q}) \\
&= (1-z)^2m^2\times\gamma^\mu \;-\; xy\times\not{q}\gamma^\mu\not{q} \\
&\quad -\; (1-z)(x-y)m\times\Big(\tfrac12\{\gamma^\mu,\not{q}\} \;=\; q^\mu\Big) \\
&\quad +\; (1-z)(x+y)m\times\Big(\tfrac12[\gamma^\mu,\not{q}] \;=\; -i\sigma^{\mu\nu}q_\nu\Big) \qquad (34)
\end{aligned}$$

where
$$(1-z)(x+y) \;=\; (1-z)^2. \qquad (35)$$

Let's plug these expressions back into eq. (29) and collect similar terms together, thus

$$\begin{aligned}
\mathcal{N}^\mu \;\cong\; & -(D-2)\,\not{\ell}\gamma^\mu\not{\ell} \;+\; 4mz(p'+p)^\mu \\
&+\; m^2\gamma^\mu\times\Big(-2-2z^2 \;+\; (4-D)(1-z)^2\Big) \\
&+\; \not{q}\gamma^\mu\not{q}\times\Big(2(z+xy) \;-\; (4-D)xy\Big) \\
&+\; mq^\mu\times(x-y)\Big(4-2z \;-\; (4-D)(1-z)\Big) \\
&+\; im\sigma^{\mu\nu}q_\nu\times\Big(2z(1+z) \;-\; (4-D)(1-z)^2\Big).
\end{aligned} \qquad (36)$$

Furthermore, in the context of the Dirac sandwich $\bar{u}(p')\Gamma^\mu u(p)$ we have

$$\not{q}\gamma^\mu\not{q} \;=\; 2q^\mu\not{q} \;-\; q^2\gamma^\mu \;\cong\; -q^2\gamma^\mu \qquad (37)$$

because $\bar{u}(p')\not{q}u(p) = 0$, and also the Gordon identity

$$(p'+p)^\mu \;\cong\; 2m\gamma^\mu \;-\; i\sigma^{\mu\nu}q_\mu\,. \qquad (38)$$

Plugging these formulae into eq. (36) and collecting similar terms, we arrive at

$$\begin{aligned}\mathcal{N}^\mu \;\cong\; & -(D-2)\,\not\ell\gamma^\mu\not\ell \;+\; m^2\gamma^\mu\times\Bigl(8z-2(1+z^2)+(4-D)(1-z)^2\Bigr)\\ &-\; q^2\gamma^\mu\times\Bigl(2(z+xy)-(4-D)xy\Bigr) - im\sigma^{\mu\nu}q_\nu\times\Bigl(4z-2z(1+z)+(4-D)(1-z)^2\Bigr)\\ &+\; mq^\mu\times(x-y)\Bigl(4-2z-(4-D)(1-z)\Bigr).\end{aligned} \tag{39}$$

To further simplify this expression, let us go back to the symmetries of the integral (22). The integral over the Feynman parameters, the integral $\int d^D\ell$, and the denominator $[l^2-\Delta]^3$ are all invariant under the parameter exchange $x\leftrightarrow y$. In eq. (39) for the numerator, the first two lines are invariant under this symmetry, but the last line changes sign. Consequently, only the first two lines contribute to the integral (22) while the third line integrates to zero and may be disregarded, thus

$$\begin{aligned}\mathcal{N}^\mu \;\cong\; & -(D-2)\,\not\ell\gamma^\mu\not\ell \;+\; m^2\gamma^\mu\times\Bigl(8z-2(1+z^2)+(4-D)(1-z)^2\Bigr)\\ &-\; q^2\gamma^\mu\times\Bigl(2(z+xy)-(4-D)xy\Bigr) \;-\; im\sigma^{\mu\nu}q_\nu\times\Bigl(4z-2z(1+z)+(4-D)(1-z)^2\Bigr)\\ =\; & -(D-2)\,\not\ell\gamma^\mu\not\ell \;+\; m^2\gamma^\mu\times\Bigl(4z\;-\;(D-2)(1-z)^2\Bigr)\\ &-\; q^2\gamma^\mu\times\Bigl(2z\;+\;(D-2)xy\Bigr) \;-\; im\sigma^{\mu\nu}q_\nu\times(1-z)\Bigl(2z+(4-D)(1-z)\Bigr).\end{aligned} \tag{40}$$

Moreover, thanks to the Lorentz invariance of the $\int d^D\ell$ integral,

$$\ell_\lambda\ell_\nu \;\cong\; g_{\lambda\nu}\times\frac{\ell^2}{D}\,, \tag{41}$$

hence

$$\not\ell\gamma^\mu\not\ell \;=\; \gamma^\lambda\gamma^\mu\gamma^\nu\times\ell_\lambda\ell_\nu \;\cong\; \gamma^\lambda\gamma^\mu\gamma^\nu\times g_{\lambda\nu}\frac{\ell^2}{D} \;=\; -(D-2)\gamma^\mu\times\frac{\ell^2}{D}\,, \tag{42}$$

and therefore

$$\begin{aligned}\mathcal{N}^\mu \;\cong\; & +\frac{(D-2)^2}{D}\times\ell^2\gamma^\mu \;+\; m^2\gamma^\mu\times\Bigl(4z\;-\;(D-2)(1-z)^2\Bigr)\\ &-\; q^2\gamma^\mu\times\Bigl(2z\;+\;(D-2)xy\Bigr) - im\sigma^{\mu\nu}q_\nu\times(1-z)\Bigl(2z+(4-D)(1-z)\Bigr).\end{aligned} \tag{43}$$

Finally, note the γ-matrix structure of the 4 terms on the RHS of eq. (43): the first 3 terms are proportional to the γ^μ, so their integrals contribute to the $F_1(q^2)$ form factor, while the fourth term is proportional to the $-i\sigma^{\mu\nu}q_\nu$ so its integral contributes to the $F_2(q^2)$. Indeed, reorganizing the numerator (43) as

$$\mathcal{N}^\mu \;\cong\; \mathcal{N}_1\times\gamma^\mu - \mathcal{N}_2\times\frac{i\sigma^{\mu\nu}q_\nu}{2m} \tag{44}$$

$$\begin{aligned}\text{where}\quad \mathcal{N}_1 &= \frac{(D-2)^2}{D}\times\ell^2 + \Big(4z-(D-2)(1-z)^2\Big)\times m^2\\ &\quad - \Big(2z+(D-2)xy\Big)\times q^2\\ &= \frac{(D-2)^2}{D}\times\ell^2 - (D-2)\times\Delta + 2z\times(2m^2-q^2)\end{aligned} \tag{45}$$

$$\text{and}\quad \mathcal{N}_2 = 2(1-z)\Big(2z+(4-D)(1-z)\Big)\times m^2 \tag{46}$$

and plugging this reorganization into the integral (22), we arrive at

$$\Gamma^\mu_{1\,\text{loop}} = F_1^{1\,\text{loop}}(q^2)\times\gamma^\mu + F_2^{1\,\text{loop}}(q^2)\times\frac{i\sigma^{\mu\nu}q_\nu}{2m}, \tag{47}$$

$$F_1^{1\,\text{loop}}(q^2) = -2ie^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_1}{\big[\ell^2-\Delta+i0\big]^3}, \tag{48}$$

$$F_2^{1\,\text{loop}}(q^2) = +2ie^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_2}{\big[\ell^2-\Delta+i0\big]^3}. \tag{49}$$

Electron's Gyromagnetic Moment

As explained earlier in class, electron's spin couples to the static magnetic field as

$$\hat{H} \supset \frac{-eg}{2m_e}\mathbf{S}\cdot\mathbf{B} \qquad \text{where}\quad g = 2\left(F_{\text{mag}} = F_1+F_2\right)\Big|_{q^2=0}. \tag{50}$$

Moreover, for $q^2=0$ the electric form factor $F_1\equiv F_{el}$ is constrained by the net charge conservation,

$$F_1^{\text{tot}} = 1^{\text{tree}} + F_1^{\text{loops}} + \delta_1 \xrightarrow[q^2\to 0]{} 1. \tag{51}$$

Therefore, the gyromagnetic moment is

$$g = 2 + 2F_2(q^2=0) \tag{52}$$

where $F_2 = F_2^{\text{loops}}$ because the there are no tree-level or counter-term contributions to the F_2, only to the F_1. Thus, to calculate the $g-2$ at the one-loop level, all we need is to evaluate the integral (49) for $q^2=0$.

Let's start with the momentum integral

$$\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_2}{\left[\ell^2-\Delta+i0\right]^3} \tag{53}$$

where $\Delta=(1-z)^2m^2$ for $q^2=0$ and $\mathcal{N}_2$ is as in eq. (46). Because the numerator here does not depend on the loop momentum ℓ, this integral converges in $D=4$ dimensions and there is no need for dimensional regularization. All we need is to rotate the momentum into Euclidean space,

$$\begin{aligned}
\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_2}{\left[\ell^2-\Delta+i0\right]^3} \;&=\; \mathcal{N}_2\times\int\frac{i\,d^4\ell_E}{(2\pi)^4}\,\frac{1}{-(\ell_E^2+\Delta)^3}\\
&=\; \frac{-i\mathcal{N}_2}{16\pi^2}\times\int\limits_0^\infty d\ell_E^2\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^3}\\
&=\; \frac{-i\mathcal{N}_2}{16\pi^2}\times\frac{1}{2\Delta}\\
&=\; \frac{-i}{32\pi^2}\times\frac{\mathcal{N}_2=4z(1-z)m^2\quad\langle\!\langle\text{for } D=4\rangle\!\rangle}{\Delta=(1-z)^2m^2\quad\langle\!\langle\text{for } q^2=0\rangle\!\rangle}\\
&=\; \frac{-i}{32\pi^2}\times\frac{4z}{1-z}\,.
\end{aligned} \tag{54}$$

Substituting this formula into eq. (49), we have

$$F_2^{1\,\text{loop}}(q^2=0) \;=\; \frac{e^2}{16\pi^2}\iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\times\frac{4z}{1-z}\,. \tag{55}$$

The integrand here depends on z but not on the other two Feynman parameters, so we can immediately integrate over x and y and obtain

$$\iint\limits_0^1 dx\,dy\,\delta(x+y+z-1) \;=\; \int\limits_0^{1-z} dx \;=\; 1-z. \tag{56}$$

Consequently,

$$F_2^{1\,\mathrm{loop}}(q^2=0) \;=\; \frac{e^2}{16\pi^2}\times\int\limits_0^1 dz\,(1-z)\times\frac{4z}{1-z} \;=\; \frac{e^2}{16\pi^2}\times 2 \;=\; \frac{\alpha}{2\pi} \tag{57}$$

and the gyromagnetic moment is

$$g \;=\; 2\,+\,\frac{\alpha}{\pi}\,+\,O(\alpha^2). \tag{58}$$

The higher-loop corrections to this gyromagnetic moment are harder to calculate because the number of diagrams grows very rapidly with the number of loops; at the 4-loop order there are thousands of diagrams, and one needs a computer just to count them! Also, at higher orders one has to include the effects of strong and weak interactions because the photons interact with hadrons and $W^\pm$ particles, which in turn interact with other hadrons, Z^0, Higgs, *etc.*, *etc.* Nevertheless, people have calculated the electron's and the muon's *anomalous magnetic moments*

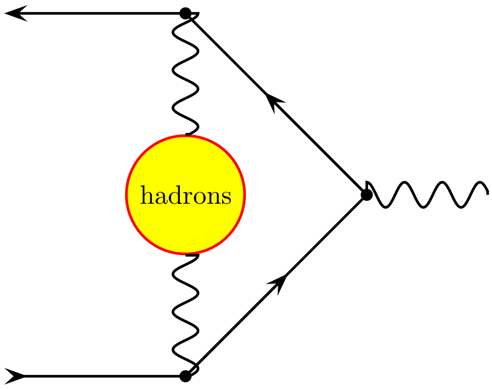
$$a_e \;=\; \frac{g_e-2}{2} \;=\; F_2^{\mathrm{electron}}(0) \quad\text{and}\quad a_\mu \;=\; \frac{g_\mu-2}{2} \;=\; F_2^{\mathrm{muon}}(0) \tag{59}$$

to the order α^4 back in the 1970s, and more recent calculations are good up to the order α^5. Meanwhile, the experimentalists have measured a_e to a comparable accuracy of 12 significant digits and a_μ to 9 significant digits

$$a_e^{\mathrm{exp}} \;=\; 0.001\,159\,652\,180\,73\;(28), \qquad a_\mu^{\mathrm{exp}} \;=\; 0.001\,165\,920\,91\;(63). \tag{60}$$

The theoretical value of the electron's anomalous magnetic moment is in good agreement with the experimental value, while for the muon there is a small discrepancy $a_\mu^{\mathrm{exp}}-a_\mu^{\mathrm{theory}}\approx(26\pm8)\cdot10^{-10}$. This discrepancy might stem from some physics beyond the Standard Model, maybe supersymmetry, maybe something else. Note that the effect of heavy particles on the a_μ is proportional to $(m_\mu/M_{\mathrm{heavy}})^2$, that's why the muon's anomalous magnetic moment is much more sensitive to the new physics than the electron's.

However, the discrepancy between the $a_\mu^{\rm exp}$ and the $a_\mu^{\rm theory}$ might also stem from a small theoretical error in modeling the photon-hadron interactions, which affects the $a_\mu^{\rm theory}$ via 2+ loop diagrams like



(61)

For a recent review of the muon's high-precision anomalous magnetic moment — both the experiments and the theory — see 2020 review of the Particle Data Group, §55, and the references cited therein.

I would like to complete this section of the notes by calculating the $F_2^{\rm 1\,loop}(q^2)$ form factor for $q^2\neq 0$. Proceeding as in eq. (54) but letting $\Delta=(1-z)^2m^2-xyq^2$, we have

$$\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_2}{\left[\ell^2-\Delta+i0\right]^3}\;=\;\frac{-i}{32\pi^2}\times\frac{4z(1-z)m^2}{(1-z)^2m^2-xyq^2} \tag{62}$$

and hence

$$F_2^{\rm 1\,loop}(q^2)\;=\;\frac{e^2}{16\pi^2}\iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\times\frac{4z(1-z)m^2}{(1-z)^2m^2-xyq^2}\,. \tag{63}$$

To evaluate this integral over Feynman parameters, we change variables from x,y,z to $w=1-z$ and $\xi=x/(x+y)$,

$$\begin{aligned} x\;=\;w\xi,\quad y\;=\;w(1-\xi),\quad z\;=\;1-w,\\ dx\,dy\,dz\,\delta(x+y+z-1)\;=\;dx\,dy\;=\;w\,dw\,d\xi. \end{aligned} \tag{64}$$

Consequently,

$$\begin{aligned}
F_2^{1\,\mathrm{loop}}(q^2) \;&=\; \frac{e^2}{16\pi^2}\int_0^1 d\xi \int_0^1 dw\, w \times \frac{4(1-w)w\times m^2}{w^2\times m^2\;-\;w^2\xi(1-\xi)\times q^2}\\
&=\; \frac{e^2}{16\pi^2}\int_0^1 d\xi \frac{m^2}{m^2\;-\;\xi(1-\xi)q^2} \times \int_0^1 dw\, w \times \frac{4w(1-w)}{w^2}\\
&=\; \frac{e^2}{8\pi^2}\times \int_0^1 d\xi \frac{m^2}{m^2\;-\;\xi(1-\xi)q^2}\\
&=\; \frac{\alpha}{2\pi}\times \frac{4m^2}{\sqrt{(-q^2)\times(4m^2-q^2)}}\times \log\frac{\sqrt{4m^2-q^2}+\sqrt{-q^2}}{2m}\\
&\quad \langle\!\langle \text{ for } q^2<0 \,\rangle\!\rangle\\
&=\; \frac{\alpha}{2\pi}\times \frac{4m^2}{\sqrt{q^2\times(4m^2-q^2)}}\times \arctan\sqrt{\frac{q^2}{4m^2-q^2}}\\
&\quad \langle\!\langle \text{ for } 0<q^2<4m^2 \,\rangle\!\rangle.
\end{aligned} \tag{65}$$

In particular, for large and spacelike q — *i.e.*, $q^2<0$ and $-q^2 \gg m^2$, —

$$F_2^{1\,\mathrm{loop}}(q^2) \;\approx\; \frac{\alpha}{2\pi}\times\frac{2m^2}{-q^2}\times\log\frac{-q^2}{m^2}\,. \tag{66}$$

The Electric Form Factor

Now consider the electric form factor $F_1(q^2)$. In the first section we have obtained

$$F_1^{1\,\mathrm{loop}}(q^2) \;=\; -2ie^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_1}{\big[\ell^2-\Delta+i0\big]^3}\,, \tag{48}$$

for

$$\mathcal{N}_1 \;\cong\; \frac{(D-2)^2}{D}\times\ell^2 \;-\; (D-2)\times\Delta \;+\; 2z\times(2m^2-q^2) \tag{45}$$

and $\Delta=(1-z)^2m^2-xyq^2$.

Let's start by calculating the momentum integral in eq. (48). The numerator $\mathcal{N}_1$ depends on ℓ as $a\ell^2 + b$, so there is a logarithmic UV divergence for $\ell \to \infty$; to regularize this divergence, we work in $D = 4 - 2\epsilon$ dimensions. Thus,

$$\begin{aligned}
-i\int\limits_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\, \frac{a\ell^2+b}{[\ell^2-\Delta+i0]^3}\ &\equiv\ -i\mu^{4-D}\int \frac{d^D\ell}{(2\pi)^D}\, \frac{a\ell^2+b}{[\ell^2-\Delta+i0]^3}\ = \\
&=\ -i\mu^{4-D}\int \frac{id^D\ell_E}{(2\pi)^D}\, \frac{-a\ell_E^2+b}{-[\ell_E^2+\Delta]^3} \\
&=\ \mu^{4-D}\int \frac{d^D\ell_E}{(2\pi)^D} \times \left[\frac{a\ell_E^2-b}{(\ell_E^2+\Delta)^3}\ =\ \frac{a}{(\ell_E+\Delta)^2}\ -\ \frac{a\Delta+b}{(\ell_E^2+\Delta)^3}\right] \\
&=\ \mu^{4-D}\int \frac{d^D\ell_E}{(2\pi)^D} \int\limits_0^\infty dt \Big(a\times t\ -\ (a\Delta+b)\times \tfrac{1}{2}t^2\Big) \times e^{-t(\Delta+\ell_E^2)} \\
&=\ \int\limits_0^\infty dt \Big(a\times t\ -\ (a\Delta+b)\times \tfrac{1}{2}t^2\Big) e^{-t\Delta} \times \mu^{4-D}\int \frac{d^D\ell_E}{(2\pi)^D}\, e^{-t\ell_e^2} \\
&=\ \int\limits_0^\infty dt \Big(a\times t\ -\ (a\Delta+b)\times \tfrac{1}{2}t^2\Big) e^{-t\Delta} \times \frac{\mu^{4-D}}{(4\pi t)^{D/2}} \\
&=\ \frac{\mu^{4-D}}{(4\pi)^{D/2}} \int\limits_0^\infty dt\, e^{-t\Delta} \times \Big(a\times t^{1-(D/2)}\ -\ \tfrac{1}{2}(a\Delta+b)\times t^{2-(D/2)}\Big) \\
&=\ \frac{\mu^{4-D}}{(4\pi)^{D/2}} \left\{a\times\Gamma\left(2-\tfrac{D}{2}\right)\times \Delta^{\frac{D}{2}-2}\ -\ \tfrac{1}{2}(a\Delta+b)\times\Gamma\left(3-\tfrac{D}{2}\right)\times\Delta^{\frac{D}{2}-3}\right\} \\
&\to\ \frac{(4\pi\mu^2)^\epsilon}{16\pi^2} \times \frac{\Gamma(1+\epsilon)}{\Delta^\epsilon} \times \left\{\frac{a}{\epsilon}\ -\ \frac{a\Delta+b}{2\Delta}\right\}.
\end{aligned} \tag{67}$$

Going back to eq. (45), we identify a and b in eq. (66) as

$$\begin{aligned}
a\ &=\ \frac{(D-2)^2}{D}\ =\ \frac{2(1-\epsilon)^2}{2-\epsilon}\,, \\
b\ &=\ 2z\times(2m^2-q^2)\ -\ (D-2)\times\Delta\ =\ \hat{b}\ -\ 2(1-\epsilon)\Delta\,,
\end{aligned} \tag{68}$$

where $\hat{b}\ =\ 2z(2m^2-q^2)$.

Consequently, on the last line of eq. (67) we have

$$\begin{aligned}
\frac{a}{\epsilon} - \frac{a\Delta+b}{2\Delta} &= a\times\left(\frac{1}{\epsilon}-\frac{1}{2}\right) + \frac{2(1-\epsilon)\Delta}{2\Delta} - \frac{\hat{b}}{2\Delta} \\
&= \frac{2(1-\epsilon)^2}{2-\epsilon}\times\frac{2-\epsilon}{2\epsilon} + (1-\epsilon) - \frac{\hat{b}}{2\Delta} \\
&= \frac{1-\epsilon}{\epsilon}\times((1-\epsilon)+\epsilon=1) - \frac{\hat{b}}{2\Delta} \\
&= \frac{1-\epsilon}{\epsilon} - \frac{z(2m^2-q^2)}{\Delta},
\end{aligned} \tag{69}$$

so the momentum integral for the electric form factors evaluates to

$$\begin{aligned}
-2ie^2\mu^{4-D}\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_1}{[\ell^2-\Delta+i0]^3} &= \\
&= \frac{\alpha}{2\pi}\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}\left\{\Gamma(\epsilon)\times(1-\epsilon) - \Gamma(1+\epsilon)\times\frac{z\times(2m^2-q^2)}{\Delta}\right\}.
\end{aligned} \tag{70}$$

The next step in our calculation is to integrate the result in eq. (70) over the Feynman parameters. Changing the integration variables from x, y, z to w and ξ according to eq. (64), we have

$$F_1^{1\,\text{loop}}(q^2) = \frac{\alpha}{2\pi}(4\pi\mu^2)^{\epsilon}\int_0^1 d\xi\int_0^1 dw\,w\times\left\{\begin{aligned} &(1-\epsilon)\Gamma(\epsilon)\times\frac{1}{[\Delta(w,\xi)]^{\epsilon}} \\ &-\Gamma(1+\epsilon)\times\frac{(1-w)(2m^2-q^2)}{[\Delta(w,\xi)]^{1+\epsilon}}\end{aligned}\right\} \tag{71}$$

where

$$\Delta(w,\xi) = (1-z)^2m^2 - xyq^2 = w^2\times\Big(m^2-\xi(1-\xi)q^2\Big), \tag{72}$$

or equivalently,

$$\Delta(w,\xi) = w^2\times H(\xi) \quad\text{where}\quad H(\xi) \stackrel{\text{def}}{=} m^2-\xi(1-\xi)q^2. \tag{73}$$

The form (73) is particularly convenient for evaluating the $\int dw$ integral in eq. (71), which becomes

$$\int_0^1 dw\left\{\frac{2(1-\epsilon)\Gamma(\epsilon)}{H^{\epsilon}}\times\frac{w}{w^{2\epsilon}} - 2\Gamma(1+\epsilon)\times\frac{2m^2-q^2}{H^{1+\epsilon}}\times\frac{w(1-w)}{w^{2+2\epsilon}}\right\}. \tag{74}$$

Near the lower limit $w\to 0$, the integrand is dominated by the second term, which is proportional to $w^{-1-2\epsilon}$. But for any $\epsilon \geq 0$ — *i.e.*, for any dimension $D \leq 4$ — the integral

$$\int_0^{\text{positive}} \frac{dw}{w^{1+2\epsilon}} \tag{75}$$

diverges: For $D = 4$ the divergence is logarithmic while for $D < 4$ it becomes power-like.

The Infrared Divergence

Physically, the divergence (75) is *infrared* rather than ultraviolet, that's why it gets worse as we lower the dimension D. Indeed, let's go back to the diagram (9) and look at the denominator $\mathcal{D}$ in eqs. (10) and (12). Taking the electron's momenta p and p' on-shell before introducing the Feynman parameters, we have

$$(p+k)^2 - m^2 = k^2 + 2kp + \cancel{p^2} - \cancel{m^2} = k^2 + 2kp = O(|k|) \quad \text{when } k \to 0, \tag{76}$$

and likewise

$$(p'+k)^2 - m^2 = k^2 + 2kp' = O(|k|) \quad \text{when } k \to 0. \tag{77}$$

Combining these two electron propagators with the $O(1/k^2)$ photon propagators, we see that the net denominator behaves as $\mathcal{D} \propto |k|^4$ for $k \to 0$ the numerator $\mathcal{N}^\mu$ remains finite, which makes the integral

$$\int d^D k \, \frac{\mathcal{N}^\mu}{\mathcal{D}} \propto \int \frac{d^D k}{|k|^4} \tag{78}$$

diverge for $k \to 0$. In $D = 4$ dimensions, the infrared divergence here is logarithmic, while in lower dimensions $D < 4$ it becomes power-like, *i.e.* $O\left((1/k_{\min})^{4-D}\right)$ — precisely as in eqs. (75) and (74).

We can regularize the infrared divergence (78) — and also (75) — by analytically continuing the spacetime dimension to $D > 4$. Such dimensional regularization of the IR divergences is used in many situations in both QFT and condensed matter. However, taking $D > 4$ makes the ultraviolet divergences worse, so if some amplitude has both UV and IR divergences, we cannot cure both of them at the same time by analytically continuing to $D \neq 4$. In particular, when calculating the electric form factor $F_1(q^2)$ of the electron, we need $D < 4$ to regulate the momentum integral $\int d^D\ell$, but then we need $D > 4$ to regulate the integral over the Feynman parameters.

A common *dirty trick* is to first continue to $D < 4$, shift the loop momentum from k^μ to $\ell^\mu = k^\mu + \text{shift}$, evaluate the $\int d^D\ell$ momentum integral in $D < 4$ dimension, then analytically continue the result to $D > 4$ and integrate over the Feynman parameters, and ultimately continue the final result to $D = 4$. However, in this kind of dimensional regularization it is hard to disentangle the $1/\epsilon$ poles coming from the UV divergence $\log(\Lambda^2/\mu^2)$ from the $1/\epsilon$ poles coming from the IR divergence $\log(\mu^2/k^2_{\rm min})$, so we are not going to use it here.

Instead, we are going to use DR for the UV divergence only, while the IR divergence is regulated by a tiny but not-quite-zero photon mass $m^2_\gamma \ll m^2_e$. Strictly speaking, a massive vector particle has three polarization states and its propagator is

$$= \frac{-i}{k^2 - m^2_\gamma + i0} \times \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2_\gamma}\right). \tag{79}$$

However, if we assume that the mass is due to Higgs mechanism and use the R_ξ gauge for $\xi = 1$ — see my notes on the Higgs mechanism, pages 6–8 for details, — then the propagator becomes simply Feynman-like

$$= \frac{-ig^{\mu\nu}}{k^2 - m^2_\gamma + i0}. \tag{80}$$

The price of an R_ξ gauge is the persistence of the Goldstone scalar — which should be eaten by the Higgs mechanism — as a scalar field with $\text{mass}^2_g = \xi m^2_\gamma$. However, this unphysical scalar field would not affect our calculations at this level of analysis.

Using this infrared regulator for the internal photon line in the one-loop diagram (9), we get the vertex amplitude that looks exactly like eq. (10) except for one factor in the denominator,

$$\frac{1}{k^2 + i0} \quad \text{becomes} \quad \frac{1}{k^2 - m^2_\gamma + i0}. \tag{81}$$

In terms of the integral (22), this change has no effect on the numerator $\mathcal{N}^\mu$, but the denominator becomes $[\ell^2 - \tilde{\Delta} + i0]^3$ where

$$\ell^2 - \tilde{\Delta}(x,y,z) = x \times ((p+k)^2 - m^2) + y \times ((p' + k)^2 - m^2) + z \times (k^2 - m^2_\gamma) \tag{82}$$

hence

$$\ell = k + xp + yp' \tag{83}$$

exactly as in eq. (20) but

$$\tilde{\Delta}(x,y,z) \;=\; (1-z)^2 m^2 \;-\; xyq^2 \;+\; z\times m_\gamma^2 \;=\; \Delta(x,y,z) \;+\; z\times m_\gamma^2\,. \tag{84}$$

Consequently, the electric form factor is

$$F_1^{1\,\text{loop}}(q^2) \;=\; \int d(FP)\int \mu^{4-D}\,\frac{d^D\ell}{(2\pi)^D}\,\frac{-2ie^2\times\mathcal{N}_1}{[\ell^2-\tilde{\Delta}+i0]^3}\,, \tag{85}$$

exactly as in eq. (48), except for the $\tilde{\Delta}$ instead of the original Δ in the denominator. The momentum integral here converges for any $D<4$ and it evaluates exactly as in eq. (67). The only subtlety here is that in the numerator, the ℓ–independent term b involves the un-modified Δ instead of Δ' (*cf.* eq. (68)), but we can fix that by writing

$$b \;=\; 2z\times\left(2m_e^2-q^2+(1-\epsilon)m_\gamma^2\right) \;-\; 2(1-\epsilon)\times\tilde{\Delta}. \tag{86}$$

Hence, instead of eq. (71) we get

$$F_1^{1\,\text{loop}}(q^2) \;=\; \frac{\alpha}{2\pi}\,(4\pi\mu^2)^\epsilon\int\limits_0^1 d\xi\int\limits_0^1 dw\,w\times\left\{\begin{array}{l} (1-\epsilon)\Gamma(\epsilon)\times\dfrac{1}{[\tilde{\Delta}(w,\xi)]^\epsilon} \\ -\,\Gamma(1+\epsilon)\times\dfrac{(1-w)(2m_e^2-q^2+(1-\epsilon)m_\gamma^2)}{[\tilde{\Delta}(w,\xi)]^{1+\epsilon}} \end{array}\right\} \tag{87}$$

where

$$\tilde{\Delta}(w,\xi) \;=\; (1-z)^2m_e^2 \;-\; xyq^2 \;+\; zm_\gamma^2 \;=\; w^2\times H(\xi) \;+\; (1-w)\times m_\gamma^2\,. \tag{88}$$

Note that the photon's mass is tiny, $m_\gamma^2 \ll m_e^2, q^2$; were it not for the IR divergences, we would have used $m_\gamma^2=0$. This allows us to neglect various $O(m_\gamma^2)$ terms in eq. (87) except when it would cause a divergence for $w\to 0$; in particular, we may neglect the $(1-\epsilon)m_\gamma^2$ term in the numerator of the second term in the integrand. As to the denominators, in eq. (88) the second term containing the photon's mass becomes important only in the $w\to 0$ limit, and in that limit

$(1-w)m_\gamma^2 \to m_\gamma^2$. Thus, we approximate

$$\tilde{\Delta}(w,\xi) \;\approx\; w^2 \times H(\xi) \;+\; m_\gamma^2 \tag{89}$$

and the $\int dw$ integral in eq. (87) becomes

$$\begin{aligned}
&\int_0^1 dw\, w \times \left\{ (1-\epsilon)\Gamma(\epsilon) \times \frac{1}{[w^2 H(\xi) + m_\gamma^2]^\epsilon} \;-\; \Gamma(1+\epsilon) \times \frac{(1-w)(2m_e^2 - q^2)}{[w^2 H(\xi) + m_\gamma^2]^{1+\epsilon}} \right\} \\
&\quad = \frac{(1-\epsilon)\Gamma(\epsilon)}{H^\epsilon} \times \int_0^1 \frac{dw\, w}{[w^2 + (m_\gamma^2/H)]^\epsilon} \\
&\qquad +\; \Gamma(1+\epsilon)\, \frac{2m_e^2 - q^2}{H^{1+\epsilon}} \times \int_0^1 \frac{dw\, w^2}{[w^2 + (m_\gamma^2/H)]^{1+\epsilon}} \\
&\qquad -\, \Gamma(1+\epsilon)\, \frac{2m_e^2 - q^2}{H^{1+\epsilon}} \times \int_0^1 \frac{dw\, w}{[w^2 + (m_\gamma^2/H)]^{1+\epsilon}}\,.
\end{aligned} \tag{90}$$

For $0 < \epsilon < \frac{1}{2}$ — *i.e.*, for $3 < D < 4$ — the integrals on the second and third lines here converge even for $m_\gamma^2 = 0$,

$$\begin{aligned}
\int_0^1 \frac{dw\, w}{[w^2]^\epsilon} &= \frac{1}{2-2\epsilon} \quad \text{for } \epsilon < 1, \\
\int_0^1 \frac{dw\, w^2}{[w^2]^{1+\epsilon}} &= \frac{1}{1-2\epsilon} \quad \text{for } \epsilon < \tfrac{1}{2},
\end{aligned} \tag{91}$$

so we may just as well evaluate them without the photon's mass. Only on the last line of eq. (90) we do need $m_\gamma^2 \neq 0$ to make the integral converge for some $D \leq 4$:

$$\int_0^1 \frac{dw\, w}{[w^2 + (m_\gamma^2/H)]^{1+\epsilon}} \;=\; \frac{-1}{2\epsilon}\; \frac{1}{[w^2 + (m_\gamma^2/H)]^\epsilon}\bigg|_0^1 \;=\; \frac{1}{2\epsilon}\left[\left(\frac{H}{m_\gamma^2}\right)^\epsilon - 1\right]. \tag{92}$$

Combining all these $\int dw$ integrals together, we get

$$\int_0^1 dw\Big\{\cdots\Big\} = \frac{\Gamma(\epsilon)}{2H^\epsilon} + \frac{\Gamma(1+\epsilon)}{1-2\epsilon}\times\frac{2m_e^2-q^2}{H^{1+\epsilon}} - \frac{\Gamma(1+\epsilon)}{2\epsilon}\times\frac{2m_e^2-q^2}{H^{1+\epsilon}}\times\left[\left(\frac{H}{m_\gamma^2}\right)^\epsilon - 1\right]$$
$$= \frac{\Gamma(\epsilon)}{2H^\epsilon}\times\left\{1 + \frac{2m_e^2-q^2}{H(\xi)}\times\left(\frac{2\epsilon}{1-2\epsilon}+1=\frac{1}{1-2\epsilon}\right) - \frac{2m_e^2-q^2}{H(\xi)}\times\left(\frac{H(\xi)}{m_\gamma^2}\right)^\epsilon\right\} \tag{93}$$

and hence

$$F_1^{1\,\mathrm{loop}}(q^2) = \frac{\alpha}{4\pi}\int_0^1 d\xi\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{H(\xi)}\right)^\epsilon\times\left\{1 + \frac{2m_e^2-q^2}{H(\xi)}\times\left[\frac{1}{1-2\epsilon} - \left(\frac{H(\xi)}{m_\gamma^2}\right)^\epsilon\right]\right\} \tag{94}$$

where

$$H(\xi) = m_e^2 - \xi(1-\xi)q^2. \tag{73}$$

Before we even try to perform this last integral, let's remember that

$$\Gamma^\mu_{\mathrm{net}} = \gamma^\mu_{\mathrm{tree}} + \Gamma^\mu_{\mathrm{loops}} + \delta_1\times\gamma^\mu \tag{5}$$

and hence

$$F_1^{\mathrm{net}}(q^2) = 1^{\mathrm{tree}} + F_1^{\mathrm{loops}}(q^2) + \delta_1\,. \tag{51}$$

Also, the net electric charge does not renormalize, so we must have

$$F_1^{\mathrm{net}}(q^2) \to 1 \quad \text{for } q^2\to 0 \quad\Longrightarrow\quad \delta_1 = -F_1^{\mathrm{loops}}(q^2=0). \tag{8}$$

In particular, the δ_1 counterterm to the order α follows from eq. (94) for $q^2=0$, in which case $H(\xi)\equiv m_e^2$ and the $\int d\xi$ integral becomes trivial (the integrand does not depend on ξ at all). Thus,

$$\delta_1 = -\frac{\alpha}{4\pi}\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon\times\left\{1+\frac{2}{1-2\epsilon} - 2\left(\frac{m_e^2}{m_\gamma^2}\right)^\epsilon\right\} + O(\alpha^2). \tag{95}$$

This formula holds for any dimension D between 3 and 4 (*i.e.*, $0<\epsilon<\frac{1}{2}$). In the $D\to 4$ limit, it becomes

$$\delta_1 \;=\; -\frac{\alpha}{4\pi}\times\left\{\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{m_e^2}\,+\,4\,-\,2\log\frac{m_e^2}{m_\gamma^2}\right\}+\;O(\alpha^2). \tag{96}$$

Now let's go back to the electric form factor $F_1^{\rm net}(q^2)$ for $q^2\neq 0$. According to eqs. (51) and (8), at the one-loop level

$$F_1^{\rm net}(q^2)\;-\;1\;=\;F_1^{\rm 1\,loop}(q^2)\;-\;F_1^{\rm 1\,loop}(0)\;+\;O(\alpha^2) \tag{97}$$

where $F_1^{\rm 1\,loop}(q^2)$ is given by eq. (94). Taking the $\epsilon\to 0$ limit of that formula, we arrive at

$$F_1^{\rm 1\,loop}(q^2)\;=\;\frac{\alpha}{4\pi}\int\limits_0^1 d\xi\left\{\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{H(\xi)}\,+\,\frac{2m_e^2-q^2}{H(\xi)}\times\left[2\,-\,\log\frac{H(\xi)}{m_\gamma^2}\right]\right\}, \tag{98}$$

and now we should subtract a similar expression for $q^2=0$. This subtraction cancels the UV divergence and the associated $1/\epsilon$ pole but not the IR divergence. Moreover, not only the subtracted one-loop amplitude depends on the IR regulators, but the coefficient of the $\log m_\gamma^2$ has a non-trivial momentum dependence. Indeed,

$$\begin{aligned}
F_1^{\rm 1\,loop}(q^2)\;-\;F_1^{\rm 1\,loop}(0)\;=&\\
=\;\frac{\alpha}{4\pi}\int\limits_0^1 d\xi&\left\{\begin{aligned}&\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{H(\xi)}\,+\,\frac{2m_e^2-q^2}{H(\xi)}\times\left[2\,-\,\log\frac{H(\xi)}{m_\gamma^2}\right]\\ &-\frac{1}{\epsilon}\,+\,\gamma_E\,-\,\log\frac{4\pi\mu^2}{m_e^2}\,-\,2\times\left[2\,-\,\log\frac{m_e^2}{m_\gamma^2}\right]\end{aligned}\right\}\\
=\;\frac{\alpha}{4\pi}\int\limits_0^1 d\xi&\left\{\log\frac{m_e^2}{H(\xi)}\,+\,\frac{2m_e^2-q^2}{H(\xi)}\times\left[2\,-\,\log\frac{m_e^2}{m_\gamma^2}\,-\,\log\frac{H(\xi)}{m_e^2}\right]\,-\,2\left[2\,-\,\log\frac{m_e^2}{m_\gamma^2}\right]\right\}\\
=\;\frac{\alpha}{4\pi}&\left[2\,-\,\log\frac{m_e^2}{m_\gamma^2}\right]\times\int\limits_0^1 d\xi\left(\frac{2m_e^2-q^2}{H(\xi)}\,-\,2\right)\\
&+\;\frac{\alpha}{4\pi}\int\limits_0^1 d\xi\,\log\frac{m_e^2}{H(\xi)}\times\left(1\,+\,\frac{2m_e^2-q^2}{H(\xi)}\right).
\end{aligned} \tag{99}$$

Following the textbook notations, let's explicitly separate the IR divergent part here from the finite part, thus

$$\begin{aligned} F_1^{\text{order}\,\alpha}(q^2) &= 1 - \frac{\alpha}{2\pi} \times \left\{ f_{\text{IR}}(q^2) \times \log\frac{m_e^2}{m_\gamma^2} + f_{\text{fin}}(q^2) \right\} + O(\alpha^2) \\ &= 1 - \frac{\alpha}{2\pi} \times \left\{ f_{\text{IR}}(q^2) \times \log\frac{-q^2}{m_\gamma^2} + \tilde{f}_{\text{fin}}(q^2) \right\} + O(\alpha^2) \end{aligned} \tag{100}$$

where

$$f_{\text{IR}}(q^2) = \int_0^1 d\xi \left(\frac{2m_e^2 - q^2}{2H(\xi)} - 1 \right), \tag{101}$$

$$f_{\text{fin}}(q^2) = \frac{1}{2} \int_0^1 d\xi \left(\frac{2m_e^2 - q^2}{H(\xi)} + 1 \right) \times \log\frac{H(\xi)}{m_e^2} - 2f_{\text{IR}}(q^2), \tag{102}$$

$$\tilde{f}_{\text{fin}}(q^2) = f_{\text{fin}}(q^2) - f_{\text{IR}}(q^2) \times \log\frac{-q^2}{m_e^2}. \tag{103}$$

All three integrals here vanish for $q^2 = 0$ — which upholds the $F_1(0) = 1$ requirement — but have finite non-zero values for all other q^2. In particular, $f_{\text{IR}}(q^2) \neq 0$, which means that the $F_1(q^2)$ form factor (100) suffers from IR divergence at all $q^2 \neq 0$.

Also, for very large q^2, the $F_1^{\text{order}\,\alpha} - 1$ grows like a quadratic polynomial in $\log(-q^2/m^2)$. Indeed, in this limit

$$f_{IR}(q^2) \approx 2\log\frac{-q^2}{m_e^2} - 1, \qquad f_{\text{fin}}(q^2) \approx \frac{1}{2}\log^2\frac{-q^2}{m_e^2} - \frac{7}{2}\log\frac{-q^2}{m_e^2} + 1 - \frac{\pi^2}{6}, \tag{104}$$

hence

$$F_1(q^2) \approx 1 - \frac{\alpha}{2\pi} \left\{ \frac{1}{2}\log^2\frac{-q^2}{m_e^2} + \left(2\log\frac{m_e^2}{m_\gamma^2} - \frac{7}{2} \right) \log\frac{-q^2}{m_e^2} - 2\log\frac{m_e^2}{m_\gamma^2} + \text{const} \right\} + O(\alpha^2). \tag{105}$$

The leading $\log^2$ term here is known as the *Sudakov's double logarithm*; it plays important role in estimating radiative corrections (*i.e.*, loop corrections) to various QED processes involving relativistic electrons. Unfortunately, I do not have class time to discuss it in any detail, but if you are interested, you may look it up in this Lipatov's review. Briefly, for any process involving high-energy electron scattering, electron-positron collisions, or leptonic decays of heavy particles, the relative magnitude of QED correction is

$$\text{correction} \;\sim\; \frac{\alpha}{\pi}\times\log^2\frac{E_e}{m_e}\,, \tag{106}$$

which becomes surprisingly large for $E_e \gtrsim 100$ GeV. And if we ever accelerate electrons to multi-TeV energies, or make particles with multi-TeV masses and observe their decays into electrons (plus some other particles), we would not be able to calculate those processes in perturbation theory because L-loop QED corrections due to Sudakov's double logarithms becomes $O(1^L)$.

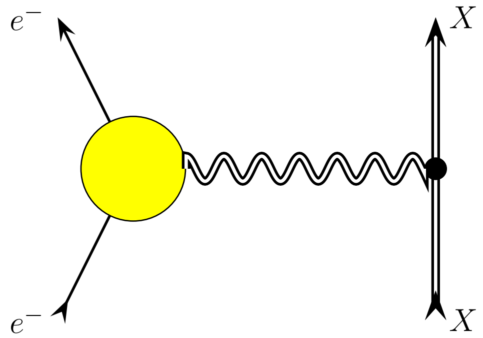
Meanwhile, let's turn our attention to a more urgent question: How does the infrared divergence of the F_1 form factor affect the physical cross-sections of various QED processes? In the next section I shall explain this issue in some detail, but let me summarize the bottom line here:

- ⋆ The S-matrix elements in QED between specific n-particle states are infrared-divergent, so the *exclusive* scattering cross-sections such as $\sigma(e^- + X \to e^- + X + \text{nothing else})$ are all infrared-divergent.
- ⋆ However, the more inclusive cross-sections such as $\sigma(e^-+X \to e^-+X+\text{optional soft photons})$ are perfectly finite: all the IR divergences cancel out. By *optional* soft photons I mean photons which would not be detected in a real-life particle experiment because their energies are too low, so we do not know if they are present or absent in the final state.

Finite Cross-sections for IR–Divergent Amplitudes

In the previous section we saw that the one-loop correction to the electron's electric form factor $F_1(q^2)$ suffers from an infrared divergence. Many other QED amplitudes — essentially, all the amplitudes involving on-shell electrons or positrons — also suffer from similar IR divergences at the one-loop and higher-loop levels. At the same time, the processes involving emission of soft (*i.e.*, low-energy) photons have IR-divergent cross-sections already at the tree level; this is explained in detail in §6.1 of the Peskin&Schroeder textbook. But somehow, the two kinds of infrared divergences cancel out from the inclusive cross-sections in which we allow for optional extra soft photons in the final state.

For example, consider elastic Coulomb scattering of an electron off a very heavy point-like particle X:



$$(107)$$

For $q^2 \ll M_X^2$, we may approximate the X particle as a static (non-recoiling) source of electric field, and in its rest frame (the lab frame), the scattering amplitude evaluates to

$$\frac{\mathcal{M}(eX \to eX)}{2M_X} \;=\; \frac{4\pi\alpha_{\rm eff}(q^2)}{q^2} \times \bar{u}(p')\Gamma^0(p',p)u(p). \tag{108}$$

At the tree level $\Gamma^\mu(p',p) \equiv \gamma^\mu$, while at the one-loop level we get non-trivial form-factors. Focusing at their infrared divergences, we have

$$F_2^{\rm net}(q^2) \;=\; \frac{\alpha}{2\pi} \times \text{finite} \;+\; O(\alpha^2) \tag{109}$$

while

$$F_1^{net}(q^2) \;=\; 1 \;-\; \frac{\alpha}{2\pi}\left(f_{\rm IR}(q^2) \times \log\frac{m_e^2}{m_\gamma^2} \;+\; \text{finite}\right) + \;O(\alpha^2) \tag{110}$$

and therefore

$$\Gamma^\mu_{\text{order}\,\alpha}(p',p) \;=\; \left(1 \;-\; \frac{\alpha}{2\pi} \times f_{\rm IR}(q^2) \times \log\frac{m_e^2}{m_\gamma^2}\right) \times \Gamma^\mu_{\rm tree} \;+\; \frac{\alpha}{2\pi} \times \text{finite}. \tag{111}$$

Plugging this one-loop vertex into the Coulomb scattering amplitude (108), we obtain

$$\mathcal{M}^{\rm tree+1\,loop}(eX \to eX) \;=\; \mathcal{M}^{\rm tree}(eX \to eX) \times \left(1 \;-\; \frac{\alpha}{2\pi} \times f_{\rm IR}(q^2) \times \log\frac{m_e^2}{m_\gamma^2} \;+\; \frac{\alpha}{2\pi} \times \text{finite}\right) \tag{112}$$

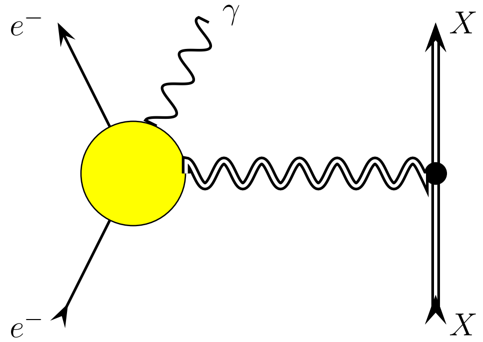
and hence the partial cross-section

$$\frac{d\sigma^{\text{tree+1 loop}}(eX\to eX)}{d\Omega} = \frac{d\sigma^{\text{tree}}(eX\to eX)}{d\Omega}\times \times\left(1-\frac{\alpha}{\pi}\times f_{\text{IR}}(q^2)\times\log\frac{m_e^2}{m_\gamma^2}+\frac{\alpha}{\pi}\times\text{finite}+O(\alpha^2)\right). \tag{113}$$

Note the IR divergence of the one-loop term in this cross-section. For future reference, I would like to rephrase it in terms of $\log(E_e/m_\gamma)$ rather than $\log(m_e/m_\gamma)$, thus

$$\frac{d\sigma^{\text{tree+1 loop}}(eX\to eX)}{d\Omega} = \frac{d\sigma^{\text{tree}}(eX\to eX)}{d\Omega}\times \times\left(1-\frac{\alpha}{\pi}\times 2f_{\text{IR}}(q^2)\log\frac{E_e}{m_\gamma}+\frac{\alpha}{\pi}\times\text{finite}+O(\alpha^2)\right). \tag{114}$$

Now consider the inelastic scattering in which a photon is emitted, $eX\to eX\gamma$,



(115)

In general, an extra incoming or outgoing photon costs a factor e in the amplitude and hence a factor α in the cross-section, thus at similar loop levels $\sigma(eX\to eX\gamma)=O(\alpha)\times\sigma(eX\to eX)$. Specifically, as explained in detail in §6.1 of the Peskin & Schroeder textbook, *for a soft photon whose energy is much smaller than the electron's, $\omega_\gamma \ll E_e$*,

$$\frac{d\sigma^{\text{tree}}(eX\to eX\gamma)}{d\Omega_e\,d\omega_\gamma} = \frac{d\sigma^{\text{tree}}(eX\to eX)}{d\Omega}\times\frac{\alpha}{\pi}\times\frac{f_{\text{IR}}(q^2)}{\omega_\gamma} \tag{116}$$ (6.25)

where the coefficient $f_{\text{IR}}(q^2)$ is precisely as in eq. (101) for the IR divergence of the F_1 form factor. I shall derive eq. (116) in the Appendix to these notes. Meanwhile, let's explore its consequences.

Integrating the partial cross-section (116) over the photon's frequencies, we immediately run into the infrared divergence:

$$\int d\omega_\gamma \, \frac{d\sigma}{d\omega_\gamma} \propto \int\limits_0^{\omega_{\max}\sim E_e} \frac{d\omega_\gamma}{\omega_\gamma} = \infty. \tag{117}$$

To regulate this divergence, we need to impose a minimal energy requirement on the emitted photon, and the simplest way to do this is to assume a tiny but non-zero photon mass m_γ. Consequently

$$\int\limits_{\text{reg}} \frac{d\omega_\gamma}{\omega_\gamma} = \log \frac{\omega_{\max}}{m_\gamma} + \text{finite} = \log \frac{E_e}{m_\gamma} + \text{finite}, \tag{118}$$

and therefore

$$\frac{d\sigma^{\text{tree}}(eX \to eX\gamma)}{d\Omega} = \frac{d\sigma^{\text{tree}}(eX \to eX)}{d\Omega} \times \frac{\alpha}{\pi} \left(2f_{\text{IR}} \times \log \frac{E_e}{m_\gamma} + \text{finite} \right). \tag{119}$$

Note that the IR divergence of this tree-level cross-section is precisely the same as of the one-loop cross-section (114), except for opposite signs,

$$\frac{d\sigma^{\text{tree+1 loop}}(Xe \to Xe)}{d\Omega} = \frac{d\sigma^{\text{tree}}(eX \to eX)}{d\Omega} \times \times \left(1 - \frac{\alpha}{\pi} \times 2f_{\text{IR}}(q^2) \log \frac{E_e}{m_\gamma} + \frac{\alpha}{\pi} \times \text{finite} + O(\alpha^2) \right), \tag{114}$$

$$\frac{d\sigma^{\text{tree}}(Xe \to Xe\gamma)}{d\Omega} = \frac{d\sigma^{\text{tree}}(eX \to eX)}{d\Omega} \times \times \left(0 + \frac{\alpha}{\pi} \times 2f_{\text{IR}}(q^2) \log \frac{E_e}{m_\gamma} + \frac{\alpha}{\pi} \times \text{finite} \right), \tag{119}$$

and therefore the combined cross-section has no infrared divergence:

$$\frac{d\sigma^{\text{tree+1 loop}}(Xe \to Xe) + d\sigma^{\text{tree}}(Xe \to Xe\gamma)}{d\Omega_e} = \frac{d\sigma^{\text{tree}}(Xe \to Xe)}{d\Omega_e} \times \left(1 + \frac{\alpha}{\pi} \times \text{finite} + O(\alpha)^2 \right). \tag{120}$$

But what do we do with the IR divergences of the partial cross-sections (114) and (119)? While it is OK to UV-regulate or IR-regulate the intermediate stages of a calculation, the final result for a measurable quantity like a partial cross-section must be finite and it cannot depend on an IR regulator like m_γ. Nevertheless, eqs. (119) and (114) seem to contradict this rule, so what gives?

To resolve this paradox, consider a real-life scattering experiment. No photon detector can detect a photon with an arbitrarily low energy ω_γ, there is always a threshold $\omega_{\rm thr} > 0$ below which the detector is blind. Thus, a final state $|Xe\gamma\rangle$ where the photon's energy is below the threshold — $\omega_\gamma < \omega_{\rm thr}$ — will be seen by the detector as simply $|Xe\rangle$ since the photon would not be detected. In other words, *observationally* we should not classify the final states by the net number of photons regardless of their energies, however low it might be. Instead, we should count the number of *detectable* photons with $\omega_\gamma > \omega_{\rm thr}$. As to the soft photons with $\omega_\gamma < \omega_{\rm thr}$, the detector would not tell us if they are there or not, so the physically measurable cross-sections should include all possible numbers of undetectably low-energy photons. In particular,

$$\begin{aligned} d\sigma(X+e\to\text{observed }X+e) &= d\sigma(X+e\to X+e)\ +\ d\sigma(X+e\to X+e+\gamma(\omega<\omega_{\rm thr}))\\ &\qquad +\ d\sigma(X+e\to X+e+\gamma(\omega<\omega_{\rm thr})+\gamma(\omega<\omega_{\rm thr}))\ +\ \cdots,\\ d\sigma(X+e\to\text{observed }X+e+\gamma) &= d\sigma(X+e\to X+e+\gamma(\omega>\omega_{\rm thr}))\\ &\qquad +\ d\sigma(X+e\to X+e+\gamma(\omega>\omega_{\rm thr})+\gamma(\omega<\omega_{\rm thr}))\ +\ \cdots,\\ &\textit{etc., etc.} \end{aligned} \tag{121}$$

To the order $O(\alpha\times\sigma^{\rm tree}(Xe\to Xe)) = O(\alpha^3)$, we should stop at one final-state photon, detectable or not, and calculate the $d\sigma(Xe\to Xe)$ to the one-loop level while the $d\sigma(Xe\to Xe\gamma)$ just to the tree level. Thus,

$$\begin{aligned} \frac{d\sigma(X+e\to\text{observed }X+e+\gamma)}{d\Omega_e} &\approx \frac{d\sigma^{\rm tree}(X+e\to X+e+\gamma(\omega>\omega_{\rm thr}))}{d\Omega_e}\\ &= \int\limits_{\omega_{\rm thr}}^{\omega_{\rm max}=O(E_e)} d\omega_\gamma\,\frac{d\sigma^{\rm tree}(X+e\to X+e+\gamma)}{d\Omega_e\,d\omega_\gamma}\\ &= \frac{d\sigma^{\rm tree}(X+e\to X+e)}{d\Omega_e}\times\frac{\alpha}{\pi}\left(2f_{\rm IR}(q^2)\log\frac{E_e}{\omega_{\rm thr}}\ +\ \text{finite}\right), \end{aligned} \tag{122}$$

where the last equality comes from

$$\frac{d\sigma^{\rm tree}(eX\to eX\gamma)}{d\Omega_e\,d\omega_\gamma} = \frac{d\sigma^{\rm tree}(eX\to eX)}{d\Omega}\times\frac{\alpha}{\pi}\times\frac{2f_{\rm IR}(q^2)}{\omega_\gamma}\qquad\langle\!\langle\text{ for }\omega_\gamma\ll E_e\,\rangle\!\rangle \tag{116}$$ (6.25)

and

$$\int\limits_{\omega_{\rm thr}}^{\omega_{\max}=O(E_e)} \frac{d\omega}{\omega} \;=\; \log\frac{E_e}{\omega_{\rm thr}} \;+\; \text{finite}. \tag{123}$$

Note that the *observed* cross-section (122) is infrared finite and does not depend on the m_γ (as long as $m_\gamma \ll \omega_{\rm thr}$). Instead, it depends on the photon detector's low-energy threshold $\omega_{\rm thr}$.

Similarly, to the same order $O(\alpha\times\sigma^{\rm tree}(Xe\to Xe) = O(\alpha^3)$,

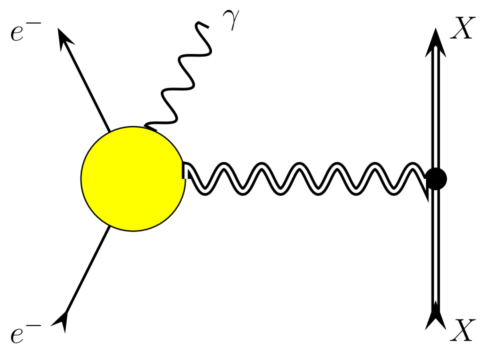
$$\begin{aligned}
&\frac{d\sigma(X+e\to\text{observed }X+e)}{d\Omega_e}\;\approx\\
&\approx\;\frac{d\sigma^{\text{tree+1 loop}}(X+e\to X+e)}{d\Omega_e}\;+\;\frac{d\sigma^{\rm tree}(X+e\to X+e+\gamma(\omega<\omega_{\rm thr}))}{d\Omega_e}\\
&\approx\;\frac{d\sigma^{\rm tree}(X+e\to X+e)}{d\Omega_e}\times\left(1-\frac{\alpha}{\pi}\times 2f_{\rm IR}(q^2)\log\frac{E_e}{\cancel{m}_\gamma}\;+\;\frac{\alpha}{\pi}\times\text{finite}\right)\\
&\quad+\;\frac{d\sigma^{\rm tree}(X+e\to X+e)}{d\Omega_e}\times\left(0+\frac{\alpha}{\pi}\times 2f_{\rm IR}(q^2)\log\frac{\omega_{\rm thr}}{\cancel{m}_\gamma}+\;\frac{\alpha}{\pi}\times\text{finite}\right)\\
&=\;\frac{d\sigma^{\rm tree}(X+e\to X+e)}{d\Omega_e}\times\left(1\;-\;\frac{\alpha}{\pi}\times 2f_{\rm IR}(q^2)\log\frac{E_e}{\omega_{\rm thr}}\;+\;\frac{\alpha}{\pi}\times\text{finite}\right).
\end{aligned}\tag{124}$$

Note how the IR regulator m_γ cancels between the two contributions to the net *observed* cross-section. Again, the observed cross-section is IR-finite, but it depends on the photon detector's threshold $\omega_{\rm thr}$.

Similar cancellations of IR divergences from the observed cross-sections happen at the higher loop orders. In general, to get a finite cross-section to the order $O(\alpha^L\times\sigma^{\rm tree})$ we should combine an L-loop cross-section with no soft photons, an $L-1$ loop cross-section with one soft photon, an $L-2$ cross-section with 2 soft photons, *etc.*, *etc.*, ending with a tree-level cross-section with L soft photons. Individually, all these formal cross-sections are infrared-divergent, but once we combine them together into a complete observed cross-section, the IR divergence should cancel out. Please read (or at least skim) textbook §6.5 to see how this works.

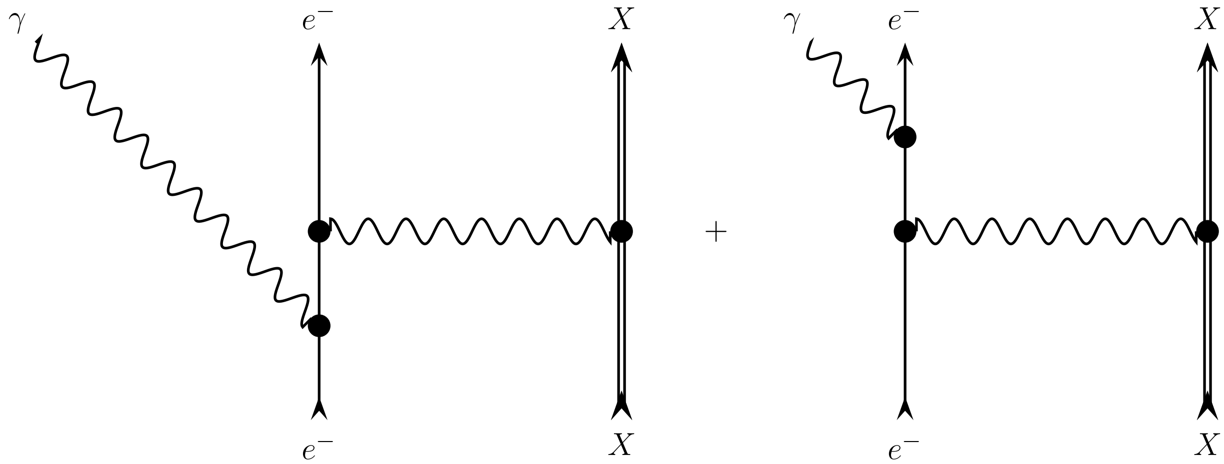
Appendix

To conclude these notes, let me derive eq. (116) for the soft-photon bremsstrahlung by the scattered electron in the



(125)

process. At the tree level, we have 2 diagrams



which evaluate to

$$\frac{\mathcal{M}_{\rm tree}}{2M_X} \;=\; \frac{e^2}{q^2}\times\left(\bar{u}(p')\gamma^0\times\frac{i}{\not{p}-\not{k}-m}(ie\,\not{\epsilon}^*)u(p) \;+\; \bar{u}(p')(ie\,\not{\epsilon}^*)\frac{i}{\not{p}'+\not{k}-m}\times\gamma^0 u(p)\right) \tag{126}$$

where ϵ^μ is the polarization vector of the outgoing photon. For the on-shell incoming electron and outgoing photons

$$(p-k)^2 \;-\; m^2 \;=\; k^2 \;-\; 2(pk) \;+\; p^2 \;-\; m^2 \;=\; -(2pk), \tag{127}$$

hence

$$
\begin{aligned}
\frac{i}{\not p - \not k - m}(ie\not\epsilon^*)u(p) &= e\frac{(\not p - \not k + m)\not\epsilon^*}{2(pk)}u(p) \\
&= e\frac{2(p\epsilon^*) + \not\epsilon^*(m - \not p) - \not k\not\epsilon^*)}{2(pk)}u(p) \\
&= e\frac{2(p\epsilon^*) - \not k\not\epsilon^*}{2(pk)}u(p) \qquad \langle\!\langle\,\text{since } (\not p - m)u(p) = 0\,\rangle\!\rangle \\
&\approx e\frac{2(p\epsilon^*)}{2(pk)}u(p) \qquad \langle\!\langle\,\text{because } k \ll p\,\rangle\!\rangle.
\end{aligned}
\tag{128}
$$

and therefore

$$
\bar{u}(p')\gamma^0 \times \frac{i}{\not p - \not k - m}(ie\not\epsilon^*)u(p) \;\approx\; \bar{u}(p')\gamma^0 u(p) \times e\frac{(p\epsilon^*)}{(pk)}\,. \tag{129}
$$

In a similar way, for the second tree diagram we get

$$
(p' + k)^2 \;=\; +2(p'k), \tag{130}
$$

hence

$$
\begin{aligned}
\bar{u}(p')(ie\not\epsilon^*)\frac{i}{\not p' + \not k - m} &= -e\bar{u}(p')\frac{\not\epsilon(\not p' + \not k + m)}{2(p'k)} \\
&= -e\bar{u}(p')\frac{2(p'\epsilon^*) + (m - p')\not\epsilon^* + \not\epsilon\not k}{2(p'k)} \\
&= -e\bar{u}(p')\frac{2(p'\epsilon^*) + \not\epsilon\not k}{2(p'k)} \qquad \langle\!\langle\,\text{since } \bar{u}(p') \times (m - \not p') = 0\,\rangle\!\rangle \\
&\approx -e\bar{u}(p') \times \frac{(p'\epsilon)}{(p'k)} \qquad \langle\!\langle\,\text{since } k \ll p'\,\rangle\!\rangle,
\end{aligned}
\tag{131}
$$

and therefore

$$
\bar{u}(p')(ie\not\epsilon^*)\frac{i}{\not p' + \not k - m} \times \gamma^0 u(p) \approx \bar{u}(p')\gamma^0 u(p) \times \frac{-(p'\epsilon^*)}{(p'k)}\,. \tag{132}
$$

Altogether, the complete tree amplitude due to both diagrams is

$$
\begin{aligned}
\mathcal{M}_{\text{tree}}(e^-X \to e^-X\gamma) &\approx \mathcal{M}_{\text{tree}}(e^-X \to e^-X) \times e\left(\frac{(p\epsilon^*)}{(pk)} - \frac{(p'\epsilon^*)}{(p'k)}\right) \\
&= \mathcal{M}_{\text{tree}}(e^-X \to e^-X) \times \frac{e}{\omega}\left(\epsilon^*_\mu\left(\frac{p'^\mu}{(np')} - \frac{p^\mu}{(np)}\right)\right)
\end{aligned}
\tag{133}
$$

where $\omega = k^0$ is the photon's frequency and n^μ is the unit 4-vector for its direction, $n^0 = |\mathbf{n}| = 1$.

To convert the amplitude (133) into a cross-section, we need to sum $|\mathcal{M}|^2$ over photon polarizations, sum/average over the electron polarizations, and take care of the phase space factors. Fortunately, the electron spin sum/average works in exactly the same way with or without the photon, thus

$$\begin{aligned}\overline{|\mathcal{M}|^2}_{\rm tree}(e^-X\to e^-X\gamma) &= \overline{|\mathcal{M}|^2}_{\rm tree}(e^-X\to e^-X)\times\frac{e^2}{\omega^2}\sum_\lambda\left|\epsilon^*_\mu(k,\lambda)\left(\frac{p'^\mu}{(np')}-\frac{p^\mu}{(np)}\right)\right|^2\\ &= \overline{|\mathcal{M}|^2}_{\rm tree}(e^-X\to e^-X)\times\frac{e^2}{\omega^2}\left[-\left(\frac{p'}{(np')}-\frac{p}{(np)}\right)^2\right].\end{aligned}\tag{134}$$

Furthermore, for $\omega\ll E_e$ the electron's phases-space factors also do not care about the extra soft photon, hence

$$d\sigma_{\rm tree}(e^-X\to e^-X\gamma) = d\sigma_{\rm tree}(e^-X\to e^-X)\times\frac{\omega^2\,d\omega\,d\Omega_n}{(2\pi)^3\,2\omega}\times\frac{e^2}{\omega^2}\left[-\left(\frac{p'}{(np')}-\frac{p}{(np)}\right)^2\right].\tag{135}$$

For high-energy electrons ($E_e\gg m_e$), the expression in $[\cdots]$ — and hence the bremmsstrahlung cross-section — is stronly peaked when the photon is emitted in the direction of the incoming or outgoung electron, $\mathbf{n}\parallel\mathbf{p}$ or $\mathbf{n}\parallel\mathbf{p}'$.

- Optional exercise for the students: Verify this, and show that each of these 2 peaks has relative height $O(E_e^2/m_e^2)$ and angular width $O(m_e/E_e)$.

But for the present purposes we are interested in the total emission rate of the soft photons rather than their angular distribution. Thus, let's average $[\cdots]$ over the photon's directions $\mathbf{n}$ and define

$$\mathcal{I}(p',p) = \int\!\frac{d^2\Omega_{\mathbf{n}}}{4\pi}\left[-\left(\frac{p'}{(np')}-\frac{p}{(np)}\right)^2\right]^{n^0=|\mathbf{n}|=1},\tag{136}$$ (6.13)

or as the textbook calls it $\mathcal{I}(\mathbf{v},\mathbf{v}')$, since it only depends on the electron's initial and final velocities but not on its mass. In terms of this $\mathcal{I}$, the cross-section relation (135) becomes

$$\frac{d\sigma^{\rm tree}(eX\to eX\gamma)}{d\Omega_e\,d\omega_\gamma} = \frac{d\sigma^{\rm tree}(eX\to eX)}{d\Omega}\times\left(\frac{e^2}{4\pi^2}=\frac{\alpha}{\pi}\right)\times\frac{\mathcal{I}(p',p)}{\omega_\gamma}\tag{137}$$ (6.25)

exactly as promised in eq. (116), except the overall coefficients is $\mathcal{I}(p',p)$ instead of $2f_{\rm IR}(q^2)$.

To conclude this appendix, we shall now prove that $\mathcal{I}(p',p) = 2f_{\rm IR}(q^2)$, so the overall coefficient in eq. (116) is indeed correct. Let's start with the integrand of eq. (136) for the $\mathcal{I}(p',p)$ and expand the Lorentz square

$$-\left(\frac{p'}{(np')} - \frac{p}{(np)}\right)^2 = -\frac{(p^2=m^2)}{(np)^2} - \frac{(p'^2=m^2)}{(np')^2} + \frac{2(pp')=2m^2-q^2}{(np)(np')}\,. \tag{138}$$

Next, we integrate each of these three terms over the directions of the unit 3-vector $\mathbf{n}$. For the first term, we have

$$\begin{aligned}\int\frac{d^2\Omega_{\mathbf{n}}}{4\pi}\,\frac{1}{(np)^2} &= \frac{2\pi}{4\pi}\int\limits_{-1}^{+1}d\cos\theta\,\frac{1}{(E-|\mathbf{p}|\times\cos\theta)^2} = \frac{1}{2}\times\frac{1}{|\mathbf{p}|}\int\limits_{E-|\mathbf{p}|}^{E+|\mathbf{p}|}\frac{d(E-|\mathbf{p}|\times\cos\theta)}{(E-|\mathbf{p}|\times\cos\theta)^2}\\ &= \frac{1}{2|\mathbf{p}|}\left(\frac{1}{E-|\mathbf{p}|} - \frac{1}{E+|\mathbf{p}|}\right) = \frac{1}{E^2-|\mathbf{p}|^2} = \frac{1}{m^2}\,.\end{aligned} \tag{139}$$

Likewise, for the second term

$$\int\frac{d^2\Omega_{\mathbf{n}}}{4\pi}\,\frac{1}{(np')^2} = \frac{1}{m^2}\,. \tag{140}$$

The third term is trickier, so let's use the Feynman parameter trick:

$$\frac{1}{(np)\times(np')} = \int\limits_0^1 d\xi\,\frac{1}{[(1-\xi)\times(np)+\xi\times(np')]^2} = \int\limits_0^1 d\xi\,\frac{1}{(np_\xi)^2} \tag{141}$$

where

$$p_\xi^\mu = (1-\xi)\times p^\mu + \xi\times p'^\mu. \tag{142}$$

Consequently,

$$\int\frac{d^2\Omega_{\mathbf{n}}}{4\pi}\,\frac{1}{(np)\times(np')} = \int\limits_0^1 d\xi\int\frac{d^2\Omega_{\mathbf{n}}}{4\pi}\,\frac{1}{(np_\xi)^2} \tag{143}$$

where

$$\int\frac{d^2\Omega_{\mathbf{n}}}{4\pi}\,\frac{1}{(np_\xi)^2} = \frac{1}{p_\xi^2} \tag{144}$$

exactly as in eqs. (139) or (140), except for $p_\xi^2 \neq m^2$ but rather

$$\begin{aligned} p_\xi^2 &= \left[(1-\xi)p + \xi p'\right]^2 \\ &= (1-\xi)^2 \times (p^2 = m^2) \; + \; \xi^2 \times (p'^2 = m^2) \; + \; \xi(1-\xi) \times (2pp' = 2m^2 - q^2) \\ &= m^2 \; - \xi(1-\xi)q^2. \end{aligned} \tag{145}$$

Altogether,

$$\int \frac{d^2\Omega_{\mathbf{n}}}{4\pi} \, \frac{1}{(np) \times (np')} \; = \int\limits_0^1 \frac{d\xi}{m^2 - \xi(1-\xi)q^2} \,. \tag{146}$$

Finally, plugging eqs. (139), (140), and (146) back into eqs. (136) and (138), we arrive at

$$\mathcal{I}(p', p) \; = \; -\frac{m^2}{m^2} \; - \; \frac{m^2}{m^2} \; + \; (2m^2 - q^2) \times \int\limits_0^1 \frac{d\xi}{m^2 - \xi(1-\xi)q^2} \; = \int\limits_0^1 d\xi \left(\frac{2m^2 - q^2}{m^2 - \xi(1-\xi)q^2} \; - \; 2 \right). \tag{147}$$

Finally comparing the last formula with the definition of the f_{IR},

$$f_{\mathrm{IR}}(q^2/m_e^2) \; = \int\limits_0^1 d\xi \left(\frac{2m_e^2 - q^2}{2[m^2 - \xi(1-\xi)q^2]} \; - \; 1 \right), \tag{101}$$

we immediately see that $\mathcal{I}(p', p) = 2 f_{\mathrm{IR}}(q^2/m_e^2)$, *quod erat demonstrandum.*