PREFACE

These notes were written for the Quantum Field Theory (II) class I taught in Spring 2021. The first section (pages 1–9) deals with imaginary ("Euclidean") time and its discretization in path integrals in the ordinary quantum mechanics, so the students in my 389 K class should be able to follow it. But mind the missing $\hbar$ factors in my formulae here, because in my QFT class I have used the $\hbar = c = 1$ units.

The remaining sections deals with subjects way beyond the scope of the Quantum Mechanics class, such as the Euclidean field theory and its Feynman rules, the StatMech / QFT analogy, and the lattice field theory. Unless you are also taking Or have already taken) a QFT class, I suggest you skip those sections.

Regulating Functional Integrals: Euclidean Time and Discretization

Formally, the Lagrangian path integral for a quantum particle in 1D is defined as

$$\iint\mathcal{D}'[\mathbf{x}(t)]\,e^{iS[\mathbf{x}(t)]} \;=\; \lim_{N\to\infty}\left(\frac{MN}{2\pi iT}\right)^{N/2}\!\int\!dx_1\cdots\!\int\!dx_{N-1}\,\exp\left(iS^{\mathrm{discr}}(x_0,x_1,\ldots,x_{N-1},x_N)\right), \tag{1}$$

but this formula raises two separate convergence problems:

(1) Convergence of the ordinary integrals $\int d^{N-1}x\,e^{iS}$ for finite N.

(2) Convergence of the continuum limit $N\to\infty$, $\Delta t\to 0$.

In these notes I deal with these convergence issues in quantum mechanics and in quantum field theory.

Let's start with the first convergence problem. In general, multi-dimensional integrals $\int d^{N-1}\vec{x}\,e^{iS(\vec{x})}$ of rapidly oscillating but unimodular functions do not converge, not even conditionally. For example, consider the Gaussian integral

$$I \;=\; \int d^D\vec{x}\,e^{i\alpha\vec{x}^2}. \tag{2}$$

In $D=1$ dimension, this integral is conditionally convergent. However, in any larger dimension $D\geq 2$, the integral becomes

$$I \;=\; \sigma(D)\times\int\limits_0^\infty dr\,r^{D-1}\times e^{i\alpha r^2}, \tag{3}$$

which is completely divergent.

However, the divergence of this integral can be regulated by means of analytic continuation. Indeed, let's analytically continue α to a complex value with a positive imaginary part. Consequently,

$$\left|e^{i\alpha\vec{x}^2}\right| \;=\; e^{-\,\mathrm{Im}(\alpha)\vec{x}^2}\;\xrightarrow[\text{rapidly}]{}\;0\quad\text{for }|\vec{x}|\to\infty\,, \tag{4}$$

which makes the integral (2) absolutely convergent. At this point, we evaluate this integral to

$$I(\alpha) \;=\; \left(\frac{\pi i}{\alpha}\right)^{D/2}, \tag{5}$$

and then we may analytically continue α back to a real value.

Similar analytic continuation regulates the divergence of the discretized path integral (1) for the discretized action

$$S^{\rm discr} \;=\; \Delta t \sum_{n=1}^{N} \left(\frac{M}{2}\left(\frac{x_n - x_{n-1}}{\Delta t}\right)^2 \;-\; V(x_n) \right). \tag{6}$$

Let's keep all the x_n real but analytically continue the time interval δt to imaginary values $\Delta t = -i\Delta t_e$ (for real and positive Δt_e). Then

$$S^{\rm discr} \;\to\; -i\Delta t_e \sum_{n=1}^{N} \left(-\frac{M}{2}\left(\frac{x_n - x_{n-1}}{\Delta t_e}\right)^2 \;-\; V(x_n) \right), \tag{7}$$

or in other words

$$iS^{\rm discr} \;\to\; -S_E^{\rm discr} \tag{8}$$

for a real and positive-definite (or at least bounded from below)

$$S_E^{\rm discr} \;=\; \Delta t_e \sum_{n=1}^{N} \left(\frac{M}{2}\left(\frac{x_n - x_{n-1}}{\Delta t_e}\right)^2 \;+\; V(x_n) \right). \tag{9}$$

Consequently, the discretized path integral

$$({\rm coeff}) \times \int d^{N-1}x \, \exp(-S_E^{\rm discr}), \tag{10}$$

becomes absolutely convergent, and once we evaluate it we may analytically continue its value back to real Δt.

In the continuum limit of integrals over paths $x(t)$, the analytic continuation $\Delta t \to -i\Delta t_e$ means *continuing to imaginary time* $t = -it_e$, although the space coordinates $x(t_e)$ remain real. Consequently, the exponent in the path integral becomes

$$\begin{aligned} iS[x(t)] &= i\int dt \left(\frac{M}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right) \\ &\longrightarrow i\int(-idt_e)\left(-\frac{M}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right) \\ &= -S_E[x(t_e)] \end{aligned} \tag{11}$$

for a real and positive definite

$$S_E[x(t_e)] = \int dt_e \left(\frac{M}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right). \tag{12}$$

Consequently, the path integral itself becomes absolutely convergent

$$\iint \mathcal{D}'[x(t_e)]\, e^{-S_e[x(t_e)]}. \tag{13}$$

(Assuming the continuum limit $N \to \infty$, $\Delta t_e \to 0$ converges, but that's a separate issue.)

In quantum field theory, the imaginary time t_e is usually called the *Euclidean time* because it acts as a fourth coordinate of a Euclidean spacetime with signature $(++++)$. That is, $x_e^\mu = (x^1, x^2, x^3, x^4) = (\mathbf{x}, t_e) = (\mathbf{x}, it)$ with metric

$$(dx_e)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 = (d\mathbf{x})^2 - (dt)^2 = -(dx^\mu\, dx_\mu)_{\text{Minkowski}}. \tag{14}$$

By extension, the action functional (12) is called the *Euclidean action*, and the path integral (13) the *Euclidean path integral.*

Going back to the real-time path integral (1), its divergence makes it ill-defined as a mathematical construct. Instead, in Physics we *define the real-time path integral as an analytic continuation from the Euclidean path integral.* Or in more detail, the path integral in real continuous time is defined though the following procedure:

1. First, we analytically continue to Euclidean time, $t \to -it_e$.

2. Second, we discretize the Euclidean time t_e by splitting it into a large number of short intervals Δt_e. In the process, the Euclidean action (12) is discretized to (9).
3. Third, we evaluate the discretized Euclidean path integral. This integral converges absolutely, but the actual evaluation may pose a challenge.
4. Fourth, we take the continuum limit, $N \to \infty$, $\Delta t_e \to 0$. In quantum mechanics this limit is usually well-behaved, but in QFT it often leads to UV divergences. I'll come back to this issue later in these notes.
5. Finally, we analytically continue the result back to the real time t.

Harmonic Oscillator Example

Let's apply the above 5-step procedure to the partition function of the harmonic oscillator. Formally,

$$\begin{aligned} Z(T) &= \operatorname{Tr}\left[e^{-iT\hat{H}} = \hat{\mathcal{U}}(T,0)\right] = \int dx_0\, U(x_0,T;x_0,0) \\ &= \int dx_0 \iint\limits_{x(0)=x_0}^{x(T)=x_0} \mathcal{D}'[x(t)]\, e^{iS[x(t)]} = \iint\limits^{x(T)=x(0)} \mathcal{D}[x(t)]\, e^{iS[x(t)]} \end{aligned} \tag{15}$$

for

$$S[x(t)] = \frac{M}{2}\int\limits_0^T dt \left(\left(\frac{dx}{dt}\right)^2 - \omega^2 x^2\right). \tag{16}$$

Analytically continuing to Euclidean time $t \to -it_e$ means also $T \to -i\beta$ and hence

$$Z(T) \to \operatorname{Tr}\left[e^{-\beta\hat{H}}\right], \tag{17}$$

which is precisely the partition function of Statistical Mechanics at temperature $= 1/\beta$. Consequently, the SM partition function obtains via Euclidean path integral

$$Z(\beta) = \iint\limits^{x(\beta)=x(0)} \mathcal{D}[x(t_e)]\, e^{-S_E[x(t_e)]} \tag{18}$$

where we integrate over *real periodic functions* $x(t_e)$ with

$$\text{period} = \beta = \frac{1}{\text{temperature}}, \tag{19}$$

and the Euclidean action in the exponent is

$$S_E[x(t_E)] \;=\; \frac{M}{2}\int_0^\beta dt\left(\left(\frac{dx}{dt_e}\right)^2 + \omega^2 x^2\right). \tag{20}$$

Next, we discretize the Euclidean time t_e by splitting the whole period β into N short intervals $\Delta t_e = \beta/N$. Consequently, the Euclidean action becomes

$$\begin{aligned} S_E^{\rm discr}(x_1,\ldots,x_N) \;&=\; \frac{M}{2}\times\frac{\beta}{N}\sum_{n=1}^N\left[\left(\frac{x_n-x_{n-1}}{\beta/N}\right)^2 \;+\;\omega^2x_n^2\right] \qquad \text{for } x_0\equiv x_N \\ &=\; \frac{NM}{2\beta}\sum_{n=1}^N\left[(x_n-x_{n-1})^2 \;+\; \frac{\omega^2\beta^2}{N^2}x_n^2\right]. \end{aligned} \tag{21}$$

Note that this discrete action is a quadratic function of the $(x_1,\ldots,x_N)$ variables, so the discretized Euclidean path integral

$$Z(\beta,N) \;=\; \left(\frac{MN}{2\pi\beta}\right)^{N/2}\int d^Nx\,\exp\bigl(-S_E^{\rm discr}(x_1,\ldots,x_N)\bigr) \tag{22}$$

is Gaussian and may be evaluated exactly. Unfortunately, the determinant of the quadratic form (21) is rather formidable, so the best way to evaluate the integral (22) is to diagonalize the action as a quadratic form.

The continuum-time Euclidean action is diagonalized via Fourier transform

$$\begin{aligned} x(t_E) \;&=\; \sum_{k=-\infty}^{+\infty}\beta^{-1/2}\,e^{-2\pi ikt_e/\beta}\times y_k\,, \\ S_E[x] \;&=\; \frac{M}{2}\sum_k\left(\omega^2\;+\;\frac{(2\pi k)^2}{\beta^2}\right)\,|y_k|^2: \end{aligned} \tag{23}$$

note that the frequencies here are discrete because the Euclidean time is periodic; also, $y_k^* = y_{N-k}$. For the discretized action (21) however, we need the discrete Fourier transform

$$x_n \;=\; \frac{1}{\sqrt{N}}\sum_{k=1}^N e^{-2\pi ikn/N}y_k \tag{24}$$

where the discrete frequencies k are defined modulo N, *i.e.* $y_0\equiv y_N$, $y_{-k}\equiv y_{N-k}$, *etc.*, *etc.*; again, the frequency modes y_k are complex, but the complete set of $y_1,\ldots y_N$ is self-conjugate as $y_k^* = y_{-k}$. The key formula of the discrete Fourier transform is

$$\sum_n e^{-2\pi i(k-\ell)n/N} \;=\; N\delta^{\mathrm{mod}\,N}(k-\ell), \tag{25}$$

which immediately leads to

$$\begin{aligned}
\sum_n^{\mathrm{mod}\,N} x_n^2 \;=\; \sum_n^{\mathrm{mod}\,N} x_n^* x_n \;&=\; \sum_n^{\mathrm{mod}\,N} \frac{1}{N} \sum_{k,\ell}^{\mathrm{mod}\,N} e^{+2\pi ikn/N} y_k^* \times e^{-2\pi i\ell n/N} y_\ell \\
&=\; \sum_{k,\ell}^{\mathrm{mod}\,N} y_k^* y_\ell \times \left(\frac{1}{N} \sum_n^{\mathrm{mod}\,N} e^{2\pi i(k-\ell)n/N} \;=\; \delta^{\mathrm{mod}\,N}(k-\ell) \right) \\
&=\; \sum_k^{\mathrm{mod}\,N} y_k^* y_k .
\end{aligned} \tag{26}$$

Also,

$$\begin{aligned}
x_n \,-\, x_{n-1} \;&=\; \frac{1}{\sqrt{N}} \sum_k^{\mathrm{mod}\,N} \left(e^{-2\pi ikn/N} - e^{-2\pi ik(n-1)/N} \right) \times y_k \\
&=\; \frac{1}{\sqrt{N}} \sum_k^{\mathrm{mod}\,N} e^{-2\pi ikn/N} \times \left(1 - e^{+2\pi ik/N} \right) \times y_k \,,
\end{aligned} \tag{27}$$

hence similarly to eq. (26),

$$\sum_n^{\mathrm{mod}\,N} (x_n - x_{n-1})^2 \;=\; \sum_k^{\mathrm{mod}\,N} \left| 1 - e^{2\pi ik/N} \right|^2 y_k^* y_k \;=\; \sum_k^{\mathrm{mod}\,N} 4\sin^2 \frac{\pi k}{N} \times y_k^* y_k \,. \tag{28}$$

Altogether, the discretized Euclidean action (21) becomes

$$S_E^{\mathrm{discr}}[y_k] \;=\; \frac{MN}{2\beta} \sum_k^{\mathrm{mod}\,N} \left(4\sin^2 \frac{\pi k}{N} \;+\; \frac{\omega^2\beta^2}{N^2} \right) |y_k|^2, \tag{29}$$

and therefore

$$Z(\beta,\omega,N) \;=\; \left(\frac{MN}{2\pi\beta} \right)^{N/2} \times J(N) \times \int d^N y \, e^{-S_E^{\mathrm{discr}}(y)} \tag{30}$$

where $J(N)$ is the Jacobian of the discrete Fourier transform (24)

$$= \left(\frac{MN}{2\pi\beta}\right)^{N/2} \times J(N) \times \prod_k^{\mathrm{mod}\,N} \sqrt{\pi} \Bigg/ \sqrt{\frac{MN}{2\beta} \times \left(4\sin^2\frac{\pi k}{N} + \frac{\omega^2\beta^2}{N^2}\right)}$$

$$= J(N) \times \prod_k^{\mathrm{mod}\,N} \left(4\sin^2\frac{\pi k}{N} + \frac{\omega^2\beta^2}{N^2}\right)^{-1/2}. \tag{31}$$

To evaluate the Jacobian J of the discrete Fourier transform, we perform the transform twice:

$$y_k = \sum_m^{\mathrm{mod}\,N} N^{-1/2} e^{-2\pi i m k/N} z_m, \qquad x_n = \sum_k^{\mathrm{mod}\,N} N^{-1/2} e^{-2\pi i k n/N} y_k = z_{-n}\,, \tag{32}$$

which immediately tells us that

$$\left(\det\left\|\frac{\partial x_n}{\partial y_k}\right\|\right)^2 = \det\left\|\frac{\partial x_n}{\partial z_m}\right\| = \pm 1.$$

Consequently, $J = |\det(\partial x_n/\partial y_k)| = 1$ and therefore

$$Z(\beta,\omega,N) = \prod_k^{\mathrm{mod}\,N} \left(4\sin^2\frac{\pi k}{N} + \frac{\omega^2\beta^2}{N^2}\right)^{-1/2}. \tag{33}$$

At this point, let me use without proof a somewhat obscure mathematical formula

$$\prod_{k=1}^{N-1} \left(2\sin\frac{\pi k}{N}\right) = N, \tag{34}$$

which allows me to re-write the discretized partition function as

$$\begin{aligned} Z(\beta,\omega,N) &= \left(0 + \frac{\omega^2\beta^2}{N^2}\right)^{-1/2} \times \prod_{k=1}^{N-1} \left(4\sin^2\frac{\pi k}{N} + \frac{\omega^2\beta^2}{N^2}\right)^{-1/2} \\ &= \frac{N}{\omega\beta} \times \prod_{k=1}^{N-1} \frac{1}{2N\sin(\pi k/N)} \times \left(1 + \frac{\omega^2\beta^2}{4N^2\sin^2\frac{\pi k}{N}}\right)^{-1/2} \\ &= \frac{1}{\omega\beta} \times \prod_{k=1}^{N-1} \left(1 + \frac{\omega^2\beta^2}{4N^2\sin^2\frac{\pi k}{N}}\right)^{-1/2}. \end{aligned} \tag{35}$$

And this is the end of step 3 — calculating the partition function in discretized Euclidean time.

The next step is taking $N \to \infty$; physically, this is the limit of continuous Euclidean time. To take this limit of eq. (35), we are going to split the product over k's into 3 sets — small $k \ll N$, small $(N-k) \ll N$, and everything in between, — and use different approximation (which become exact for $N \to \infty$) for each set. Specifically:

$$\text{for} \quad 1 < k \ll N: \quad 4N^2 \sin^2 \frac{\pi k}{N} \approx (2\pi k)^2, \tag{36}$$

and likewise

$$\text{for} \quad 1 < (N-k) \ll N: \quad 4N^2 \sin^2 \frac{\pi k}{N} \approx (2\pi(N-k))^2, \tag{37}$$

while for the remaining modes

$$4N^2 \sin^2 \frac{\pi k}{N} \gg 1 \quad \Longrightarrow \quad 1 + \frac{\omega^2 \beta^2}{4N^2 \sin^2 \frac{\pi k}{N}} \approx 1. \tag{38}$$

Consequently,

$$\begin{aligned} Z(\beta, \omega, N) \underset{N \gg 1}{\longrightarrow} & \frac{1}{\omega\beta} \times \prod_{1 \leq k \ll N} \left(1 + \frac{\omega^2\beta^2}{(2\pi k)^2}\right)^{-1/2} \times \prod_{1 \leq (N-k) \ll N} \left(1 + \frac{\omega^2\beta^2}{(2\pi(N-k))^2}\right)^{-1/2} \\ &= \frac{1}{\omega\beta} \times \prod_{1 \leq k \ll N} \left(1 + \frac{\omega^2\beta^2}{(2\pi k)^2}\right)^{-1} \\ & \underset{N \to \infty}{\longrightarrow} \frac{1}{\omega\beta} \times \prod_{k=1}^{\infty} \left(1 + \frac{\omega^2\beta^2}{(2\pi k)^2}\right)^{-1}. \end{aligned} \tag{39}$$

To evaluate the infinite product on the bottom line of eq. (39), consider it as an analytic function of a complex variable ν evaluated for $\nu = \omega\beta/2$,

$$f(\nu) = \frac{1}{2\nu} \times \prod_{k=1}^{\infty} \frac{1}{1 + (\nu/\pi k)^2}, \qquad Z(\omega, \beta) = f(\nu = \omega\beta/2). \tag{40}$$

The infinite product here converges absolutely for all finite $\nu \in \mathbf{C}$, so $f(\nu)$ is an analytic function whose only singularities are poles and zeros. In fact, it has no zeros while its poles are located along the imaginary axis at $\nu = \pi i \times$ an integer; indeed,

$$2f(\nu) \;=\; \frac{1}{\nu} \times \prod_{k=1}^{\infty} \frac{(\pi k)^2}{(\nu - \pi i k) \times (\nu + \pi i k)}\,. \tag{41}$$

In other words, $2f(\nu)$ has the same poles and zeros as the $1/\sinh(\nu/2)$ function, and indeed there is a well-known formula

$$\sinh(\nu) \;=\; \nu \times \prod_{k=1}^{\infty} \bigl(1 + (\nu/\pi k)^2\bigr) \tag{42}$$

hence

$$2f(\nu) \;=\; \frac{1}{\sinh(\nu)}\,. \tag{43}$$

Thus, in the continuous Euclidean time limit we get

$$Z(\beta,\omega) \;=\; \frac{1}{2\sinh(\beta\omega/2)}\,. \tag{44}$$

Finally, analytically continuing back to the real time, we get

$$Z(T,\omega) \;=\; \frac{1}{2i\sin(\omega T/2)}\,. \tag{45}$$

Either of these partition functions can be used to obtain the energy spectrum of the quantum harmonic oscillator. For example,

$$\begin{aligned} Z(\beta,\omega) \;&=\; \frac{1}{2\sinh(\omega\beta/2)} \;=\; \frac{1}{e^{+\omega\beta/2} \,-\, e^{-\omega\beta/2}} \\ &=\; e^{-\omega\beta/2} \times \bigl(1 - e^{-\omega\beta}\bigr)^{-1} \;=\; e^{-\omega\beta/2} \times \sum_{n=1}^{\infty} e^{-n\omega\beta} \\ &=\; \sum_{n=1}^{\infty} e^{-(n+\frac{1}{2})\omega\beta}, \end{aligned} \tag{46}$$

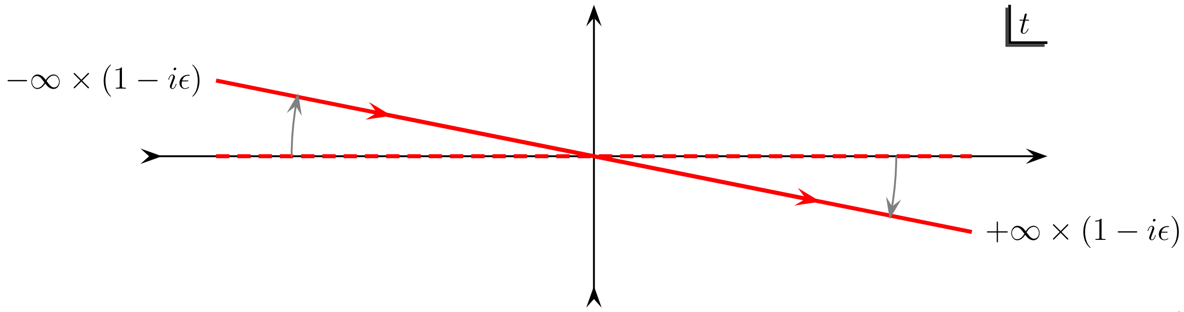
hence non-degenerate eigenvalues $E_n = (n + \frac{1}{2})\omega$ for $n = 0, 1, 2, \ldots$.

QFT Functional Integral in Euclidean Spacetime

When we calculated the n-field correlation functions

$$G_n(x_1,\ldots,x_n) \;=\; \langle\Omega|\,\mathbf{T}\,\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle \tag{47}$$

back in January (*cf.* my notes), we had tilted the time axis in the complex plane

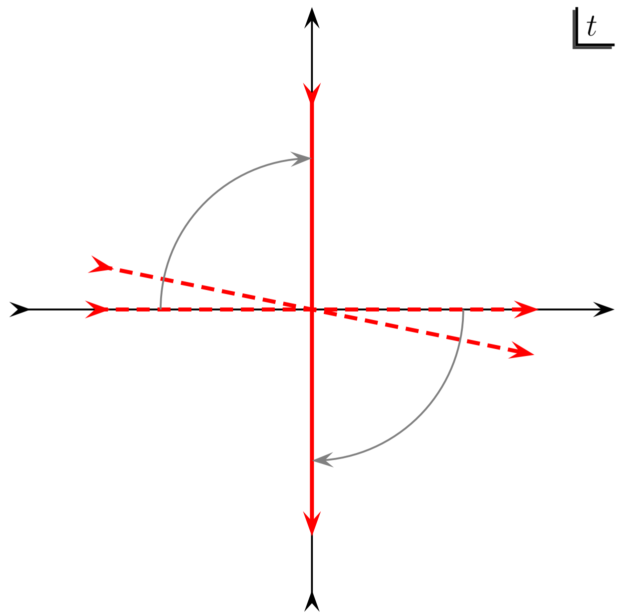


(48)

in order to project the initial and the final states of the matrix element onto the vacuum state of the theory. In order to define the "path" integrals for the quantum fields such as

$$Z[J] \;=\; \iint\mathcal{D}[\Phi(\mathbf{x},t)]\,\exp\left(i\int d^4x\,(\mathcal{L}+J\Phi)\right), \tag{49}$$

we need to analytically continue from the real time $t=x^0$ to imaginary time $t=-it_e$; in terms of the complex time diagram (48) this means tilting the time axis the whole 90° till it points straight down,



(50)

After this tilt, $t = -it_e$ becomes imaginary (for real t_e) while the space coordinates (x^1, x^2, x^3) remain real, and the scalar field values $\Phi(\mathbf{x}, t_e)$ remain real. From the spacetime point of view, we may identify the t_e as x^4, the fourth coordinate of a Euclidean 4D spacetime with positive metric

$$x_e = (\mathbf{x}, x^4 = t_e), \qquad (dx_e)^2 = (d\mathbf{x})^2 + (dx^4)^2 = (d\mathbf{x})^2 - (dt)^2 = -(dx^\mu dx_\mu)_{\text{Minkowski}}. \tag{51}$$

This Euclidean spacetime has $SO(4)$ rotational symmetry, which is the analytic continuation of the $SO^+(3,1)$ Lorentz symmetry of the Minkowski spacetime.

In Euclidean spacetime $\partial_0 = \partial/\partial t$ becomes $+i\partial_4 = +i\partial/\partial t_e$, in perfect agreement with $p^0 = +ip^4$ for Minkowski vs. Euclidean momenta. Thanks to $\partial_0 = i\partial_4$, the kinetic term in the scalar field's Lagrangian becomes

$$\tfrac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) = \tfrac{1}{2}(\partial_0\Phi)^2 - \tfrac{1}{2}(\nabla\Phi)^2 = -\tfrac{1}{2}(\partial_4\Phi)^2 - \tfrac{1}{2}(\nabla\Phi)^2 = -\tfrac{1}{2}(\partial_\mu\Phi)_e^2, \tag{52}$$

hence

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - V(\Phi) \quad\longrightarrow\quad -\mathcal{L}_E \\ \text{for } \mathcal{L}_E &= +\tfrac{1}{2}(\partial_\mu\Phi)_E^2 + V(\Phi), \end{aligned} \tag{53}$$

and therefore

$$\begin{aligned} iS[\Phi(x), J(x)] &= i\int d^4x\,\big(\mathcal{L} + J\Phi\big) \quad\longrightarrow\quad -S_E[\Phi(x_e), J(x_e)] \\ \text{for } S_E[\Phi(x_e), J(x_e)] &= \int d^4x_e\,\big(\mathcal{L}_E - J\Phi\big). \end{aligned} \tag{54}$$

Consequently, the Euclidean functional integral

$$Z[J(x_e)] = \iint \mathcal{D}[\Phi(x_e)]\,\exp\left(-\int d^4x_e(\mathcal{L}_E - J\Phi)\right) \tag{55}$$

converges absolutely, so all we have to worry about are the discretization and the eventual continuum limit.

In light of the signs in eq. (55) and the absence of any imaginary factors, in the Euclidean spacetime the connected correlation functions obtain from variational derivatives of $\log Z[J]$ without any factors of i, thus

$$G_n^{\rm conn}(x_1,\ldots,x_n) \;=\; \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\,\log Z[J]\bigg|_{J\equiv 0}\,. \tag{56}$$

At the the connected Feynman diagrams contributing to these correlation functions, we are used to calculating them by starting with the Minkowski-space Feynman rules, but then analytically continuing to the Euclidean loop momenta to calculate the integrals. But now we can take a simpler way by using the Euclidean Feynman rules to begin with. Starting from the Euclidean Lagrangian

$$\mathcal{L}_E \;=\; \mathcal{L}_E^{\rm free} \,+\, \mathcal{L}_E^{\rm pert}, \quad \mathcal{L}_E^{\rm free} \;=\; \frac{1}{2}(\partial_\mu\Phi)_E^2 \,+\, \frac{m^2}{2}\Phi^2 = \frac{1}{2}\Phi(m^2-\partial_e^2)\Phi, \quad \mathcal{L}_E^{\rm pert} \;=\; +\frac{\lambda}{24}\Phi^4 \tag{57}$$

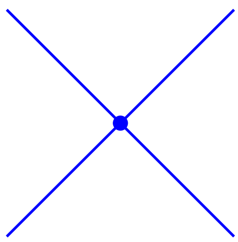
and the perturbative expansion of the Euclidean path integral as

$$\iint\!\mathcal{D}[\Phi(x)]\,\exp\left(-\int\! d^4x_e\,\mathcal{L}_E^{\rm free}\right)\times\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\int\! d^4x_e\,\mathcal{L}_E^{\rm pert}\right)^n, \tag{58}$$

the propagator obtains from the free Lagrangian as

$$= \;\frac{1}{m^2-\partial_e^2} \;=\; \frac{1}{m^2+p_e^2} \tag{59}$$

without any factors of i, and the vertex follows from the perturbation $\mathcal{L}_E^{\rm pert}$ and the red minus sign in the expansion (58),



$$= \;-\lambda. \tag{60}$$

Note: the vertex factor also does not have an i factor, but it carries and overall minus sign (relative to perturbation $\mathcal{L}_E^{\rm pert}$) stemming from the minus sign in $\exp(-S_E)$.

QFT / StatMech Analogy

The Euclidean path integral formulation of the quantum field theory is rather similar to the equilibrium statistical mechanics of condensed matter systems. For an example, consider a magnetic material in an external magnetic field $\mathbf{B}(\mathbf{x})$. Microscopically, the magnetization stems from atomic spins and hence magnetic moments, but macroscopically — *i.e.*, at distance scales much larger that distances between neighboring atoms, — we may describe the magnetization by a classical macroscopic field $\mathbf{M}(\mathbf{x})$. For simplicity, let's assume a preferred magnetization axis (and the external field parallel to that axis), so we may treat the magnetization field as a scalar rather than a vector.

The classical Hamiltonian of the magnetization field has general form

$$\begin{aligned} H[M(\mathbf{x})] &= \int d^3\mathbf{x}\left(\frac{a}{2}(\nabla M)^2 + V(M) - BM\right) \\ \text{where}\quad V(M) &= \frac{b}{2}M^2 + \frac{c}{24}M^4 + \cdots \end{aligned} \tag{61}$$

for some constants $a, b, c, \ldots$. Consequently, the thermal equilibrium at temperature T of the magnetization field is governed by the partition function

$$Z[B(\mathbf{x})] = \iint \mathcal{D}[M(\mathbf{x})]\, e^{-H[M(\mathbf{x})]/T} \tag{62}$$

where

$$\iint \mathcal{D}[M(\mathbf{x})] = \begin{pmatrix}\text{continuum}\\ \text{limit of}\end{pmatrix}\left(\begin{pmatrix}\text{normalization}\\ \text{factor}\end{pmatrix} \times \prod_{\text{atoms}} \int dm_{\text{atom}}\right). \tag{63}$$

The Helmholtz's free energy follows from the partition function as

$$F[B(\mathbf{x})] = -T\log Z[B(\mathbf{x})] \tag{64}$$

and then acts as a generation functional of the connected correlation functions of the magnetization field,

$$G_n^{\text{conn}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \stackrel{\text{def}}{=} \langle M(\mathbf{x}_1)\cdots M(\mathbf{x}_n)\rangle^{\text{conn}} = -T^{n-1}\frac{\delta F[B(\mathbf{x})]}{\delta B(\mathbf{x}_1)\cdots\delta B(\mathbf{x}_n)}. \tag{65}$$

Altogether, we see a clear analogy between the classical magnetization field $M(\mathbf{x})$ in 3D statistical mechanics and the quantum field $\Phi(x_e)$ in 4D Euclidean spacetime. The integral (63) over the magnetization field is the obvious analogy of the functional integral over

$\Phi(x_e)$, the classical Hamiltonian (61) corresponds to QFT's Euclidean action

$$S_E[\Phi(x)] \;=\; \int d^4x_e\left(\tfrac{1}{2}(\partial_\mu\Phi)^2_e \;+\; V(\Phi) \;-\; J\Phi\right), \tag{66}$$

the external magnetic field $B(\mathbf{x})$ corresponds to the source $J(x_e)$, the statistical partition function (62) — to the QFT partition function $Z[J(x_e)]$, and the the statistical correlation functions (65) — to the quantum correlation functions of the QFT.

Likewise, the Euclidean path integral formulations of more general quantum field theories make them analogous to *classical* statistical mechanics in 4D.

In this analogy, the Euclidean action of QFT plays the role of the classical Hamiltonian of a StatMech system, which begs the question: *What is the QFT's analogue of the temperature T?* At first blush, the answer seems to be the Planck's constant $\hbar$; indeed, the StatMech partition function (62) is the integral of $\exp(-H/T)$ while the QFT's partition function is the integral of $\exp(-S_E/\hbar)$. However, $\hbar$ is a universal constant — which can be set to 1 by a choice of units — so it cannot vary like we can vary the temperature of a condensed matter. Instead, the proper QFT analog of the temperature is the coupling constant, such as λ, α_{QED}, or α_{QCD}.

To see how this works in the $\lambda\phi^4$ theory, let us rescale rescale the $\Phi(x)$ field by the factor $1/\sqrt{\lambda}$:

$$\Phi(x) \;=\; \frac{1}{\sqrt{\lambda}}\,\varphi(x), \tag{67}$$

hence

$$\frac{1}{2}(\partial_\mu\Phi)^2_e \;=\; \frac{1}{2\lambda}(\partial_\mu\varphi)^2_e,\quad \frac{m^2}{2}\,\Phi^2 \;=\; \frac{m^2}{2\lambda}\,\varphi^2,\quad \frac{\lambda}{24}\,\Phi^4 \;=\; \frac{1}{24\lambda}\,\varphi^4, \tag{68}$$

and therefore

$$\begin{aligned} S_E[\Phi(x)] \;&=\; \int d^4x_e\left(\frac{1}{2}(\partial_\mu\Phi)^2_e \;+\; \frac{m^2}{2}\,\Phi^2 \;+\; \frac{\lambda}{24}\Phi^4\right) \\ &=\; \frac{1}{\lambda}\int d^4x_e\left(\frac{1}{2}(\partial_\mu\varphi)^2_e \;+\; \frac{m^2}{2}\,\varphi^2 \;+\; \frac{1}{24}\,\varphi^4\right) \\ &=\; \frac{1}{\lambda}\times H[\varphi(x)]. \end{aligned} \tag{69}$$

Thus, once we map the rescaled field $\varphi(x)$ (rather than the original field $\Phi(x)$) onto the magnetization field $M(x)$ of statistical mechanics, then $H[\varphi(x)]$ maps onto the classical Hamiltonian for the $M(x)$ while λ maps onto the temperature T so that

$$\exp(-S_E) \;=\; \exp(-H[\varphi]/\lambda) \;\longleftrightarrow\; \exp(-H[M]/T). \tag{70}$$

Likewise, in the Yang–Mills theory we simply use the group-normalized (rather than canonically normalized) gauge fields $\mathcal{A}_\mu(x) = gA_\mu(x)$ and tensions

$$\mathcal{F}_{\mu\nu}(x) \;=\; gF_{\mu\nu}(x) \;=\; \partial_\mu\mathcal{A}_\nu \;-\; \partial_\nu\mathcal{A}_\mu \;+\; i[\mathcal{A}_\mu,\mathcal{A}_\nu] \tag{71}$$

so the Euclidean Yang–Mill action becomes

$$S_E[\mathcal{A}^\mu(x)] \;=\; \frac{1}{2g^2}\int\! d^4x_e\ \mathrm{tr}(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}). \tag{72}$$

Consequently, the overall factor $1/g^2$ plays the role of $1/T$, so g^2 — or equivalently $\alpha = g^2/4\pi$ — plays the temperature's role.

In both examples, a strongly coupled QFT acts as a hot condensed matter where the fluctuations explore most of the system's phase space. On the other hand, a weakly coupled QFT acts as a cold condensed matter which sticks to the lowest-energy configurations and *small* fluctuations around them. The lowest-energy configurations here are the minima of the Hamiltonian functional; in QFT terms, they are the minima of the action and hence solutions of the classical field equations. And the small fluctuations around these solutions are governed by the perturbation theory.

Lattice Field Theory

In condensed matter, $M(\mathbf{x})$ is the macroscopic field; microscopically, there are magnetic moments of individual atoms, thus

$$\iint\!\mathcal{D}[M(\mathbf{x})] \;=\; \begin{pmatrix}\text{continuum}\\ \text{limit of}\end{pmatrix}\left(\begin{pmatrix}\text{normalization}\\ \text{factor}\end{pmatrix}\times\prod_{\text{atoms}}\int\! dm_{\text{atom}}\right). \tag{63}$$

Moreover, the atoms form a discrete crystalline lattice rather than a continuous space.

By analogy, in quantum field theory the proper definition of the path integral involves discretizing all 4 dimensions of the Euclidean spacetime, *i.e.* replacing the 4 continuous coordinates x_e^μ with some kind of a 4D crystalline lattice. The simplest such lattice is hypercubic:

$$x_e^\mu \;=\; an_e^\mu \;=\; (an^1, an^2, an^3, an^4) \quad \text{for integer} \quad n^1, n^2, n^3, n^4 \in \mathbf{Z}. \tag{73}$$

The a here is the *lattice constant* which we take to be very short. On the lattice, the scalar field $\Phi(x_e)$ becomes a discrete set of variables $\Phi(n_e)$, one variable for each lattice site (73), hence the path integral becomes the product of ordinary integrals,

$$\iint \mathcal{D}[\Phi(x_e)]\, e^{-S_E[\Phi]} \;\longrightarrow\; \begin{pmatrix}\text{normalization}\\ \text{factor}\end{pmatrix} \times \prod_{n_e} \int d\Phi(n_e)\, \exp\Bigl(-S_E^{\rm discr}\bigl(\text{all the } \Phi(n_e)\bigr)\Bigr). \tag{74}$$

As to the discretized Euclidean action, we use

$$\frac{1}{a}\Bigl(\Phi(n_e + 1_\mu) \;-\; \Phi(n_e)\Bigr) \;\underset{a\to 0}{\longrightarrow}\; \partial_\mu \Phi(x_e) \tag{75}$$

(where 1_μ denotes the unit vector in the Euclidean direction μ), so we may use the LHS here as the definition of the derivative in the lattice space theory, hence,

$$S_E^{\rm discr} \;=\; a^4 \sum_{n_e} \left(\frac{1}{2a^2} \sum_\mu \bigl(\Phi(n_e + 1\mu) - \Phi(n_e)\bigr)^2 \;+\; \frac{m^2}{2}\Phi^2(n_e) \;+\; \frac{\lambda}{24}\Phi^4(n_e) \;-\; J(n_e)\Phi(n_e) \right) \tag{76}$$

where the overall factor a^4 comes from discretizing the spacetime integral $\int d^4x_e$. Altogether, the partition function of the lattice field theory is

$$Z[J(n_e)] \;=\; \begin{pmatrix}\text{normalization}\\ \text{factor}\end{pmatrix} \times \prod_{n_e} \int d\Phi(n_e)\, \exp\bigl(-S_E^{\rm discr}[\Phi(n_e), J(n_e)]\bigr) \tag{77}$$

where all the integrals are absolutely convergent.

From the low-energy point of view, the lattice is a non-perturbative ultraviolet cutoff $\Lambda = \pi/a$. To see how this works, let's start with a periodic lattice in 1 dimension of space. The momentum space for such a lattice is a periodic circle of length $2\pi/a$; indeed

$$\text{for any } p' \;=\; p \;+\; (2\pi/a) \times \text{integer}, \quad e^{ip'x} \;=\; e^{ipx} \text{ for any } x \in \text{lattice}, \tag{78}$$

hence p' is equivalent to p. Likewise, for the hypercubic lattice in 4 Euclidean dimensions, each component p_e^μ of any Euclidean momentum spans a periodic circle of length $2\pi/a$. so we may parametrize it by $-(\pi/a) \leq p_e^\mu \leq +(\pi/a)$ only. Consequently, the momentum space has a finite volume $(2\pi/a)^4$ similar to the hard-edge UV cutoff, except that there is no actual edge: Instead of a ball with a hard spherical surface, the lattice momentum space is topologically a 4D torus T^4 — a direct product of 4 circles, — and it does not have any surface. This allows us to shift the loop momentum variables by constants when evaluating the Feynman diagrams. However, the flip side of the T^4 momentum-space geometry is the rather ugly scalar propagator:

$$\begin{aligned} \underline{\hspace{3cm}} \quad &= \left(m^2 + \frac{4}{a^2}\sum_\mu \sin^2(ap^\mu/2)\right)^{-1} \\ &\longrightarrow \frac{1}{m^2+p_e^2} \quad \textbf{only} \text{ for } p_e^2 \ll (1/a)^2 \end{aligned} \tag{79}$$

which makes the lattice a rather inconvenient cutoff for the perturbation theory.

On the other hand, the lattice field theory allows non-perturbative calculation of path integrals like (77) on a computer. In practice, this means simulating the statistical mechanics of the 4D classical lattice theory with probability distribution $\exp(-S_E^{\text{discr}})$ using a Monte–Carlo algorithm such as Metropolis–Hasting. For some theories, there are other non-perturbative methods for calculating the functional integrals of lattice theories, for example the strong-coupling expansion in powers of $1/g^2$ for $g^2 \gg 1$. However, such methods go way beyond the scope of this class.

Note that the parameters λ and m^2 of the discretized action (76) are the *bare coupling* λ_b and the *bare mass*^{2} m_b^2 for the lattice cutoff rather than the physical coupling or mass^2. Likewise, the lattice field $\Phi(x_e = an_e)$ is the bare field $\Phi_b(x_e) = \sqrt{Z}\Phi_{\text{ren}}(x_e)$ rather than the renormalized field. Consequently, when we change the lattice spacing a, we must adjust the λ_b and m_b^2 parameters and the field strength factor $\sqrt{Z}$ in order to keep the long-distance physics invariant. In particular, we should adjust the bare mass such that the physical mass comes out much smaller than than the lattice cutoff, $m_{\text{phys}}^2 \ll (1/a)^2$, otherwise the scalar field would not propagate to macroscopic distances $\gg a$. In condensed matter terms, this corresponds to keeping the dimensionless bare parameters $a^2 m_b^2$ and λ_b very close to a critical point.

Lattice Symmetries

The continuous Euclidean spacetime has an $SO(4)$ rotation symmetry, the analytic continuation of the $SO^+(3,1)$ Lorentz symmetry of the Minkowski spacetime. The discrete lattice has only a discrete group of symmetries such as the hypercubic group $\mathbf{HC} = SO(4;\mathbf{Z})$ subgroup of the 4D rotation group $SO(4)$, so the geometric symmetries of the lattice field theory are also limited to the hypercubic group. Indeed, the lattice propagator (79) is invariant under the hypercubic symmetries of the momentum vector p_e^μ but not under general $SO(4)$ symmetries. Consequently, in the continuum limit of the lattice field theory, its Euclidean Lagrangian contains all kinds of terms which respect the hypercubic symmetry but break the $SO(4)$, for example

$$\mathcal{L}_E \supset \frac{c}{2}\sum_\mu \left(\partial_\mu^2\Phi\right)^2 + \frac{c'}{2}\sum_{\mu\nu}(\partial_\mu\partial_\nu\Phi)^2 \tag{80}$$

for some un-equal couplings $c' \neq c$. (For $c' = c$ this operators would be $SO(4)$ invariant, but the hypercubic symmetry allows for $c' \neq c$.) Fortunately, all such operators have dimensions $\Delta \geq 6 > 4$ which make them *irrelevant* from the RG point of view. Indeed, in the continuum limit of a lattice theory, any operator of dimension $\Delta > 4$ has a coupling of magnitude no stronger than $O(a^{\Delta-4})$, — for example, the operators (80) have couplings $c = -a^2/12$ and $c' = 0$, — so the dimensionless strength of such an operator at energy $E \ll (1/a)$ is

$$O\left((Ea)^{\text{positive}}\right) \longrightarrow 0 \quad \text{for} \quad E \to 0. \tag{81}$$

Consequently, at energies *much* lower than the lattice cutoff, all such operators have negligibly weak couplings so we may simply disregard them. Thus, at low-energies we may limit the effective Lagrangian of the renormalized theory to the relevant and marginal operators only. Fortunately, all relevant or marginal operators which respect the hypercubic symmetry of the lattice also respect the $SO(4)$ rotational symmetry, and that's how the continuum limit of the lattice field theory becomes $SO(4)$ symmetric, hence Lorentz symmetric in the Minkowski spacetime.

This mechanism is a special case of a *custodial symmetry*. In general, it works like this: Suppose some QFT has an exact symmetry group G. It can be global or local, discrete or continuous, spacetime or internal, it does not matter as long as it's exact. Now write down all the relevant and marginal operators (*i.e.*, operators of dimension $\Delta \le D$) which are invariant under G. Sometimes, all such operators would be invariant under a bigger symmetry group $H \supset G$. In this case, G acts as a custodial symmetry of H: it limits H violation to irrelevant operators with very small low-energy couplings.

For a non-geometric example, consider baryon number violation in the Standard Model. Thanks to Lorentz and $SU(3)_{\text{color}}$ symmetries of the Standard Model, any B-violating operator has to have dimension $\Delta \ge 6$, for example

$$\epsilon_{ijk}\Psi^i_{\text{quark}}\Psi^j_{\text{quark}}\Psi^k_{\text{quark}}\Psi_{\text{lepton}} \tag{82}$$

(never mind the Dirac and flavor indices). At low energies, the dimensional strength of such an operator is suppressed by powers of $(E/\Lambda)^{\Delta-4\ge 2}$ where Λ is the upper limit on energies for which we may use the Standard Model, hence proton decay rate is limited to

$$\Gamma \lesssim \frac{M_p^5}{\Lambda^4}\,. \tag{83}$$

In particular, for $\Lambda \gtrsim 10^{16}$ GeV, we get the proton lifetime within the current experimental limit $(1/\Gamma) > 10^{34}$ year.

On the other hand, the experimental limit on proton decays is so stringent that even an irrelevant operators can cause trouble if its coupling is greater than 10^{-32} $(\text{GeV})^{4-\Delta}$. Avoiding such operators is a major constraint on beyond-the-Standard-Model physics, especially on lepto-quarks and other colored particles of masses $M \lesssim 10^{16}$ GeV. In particular, this constraint rules out several Grand Unified Theories such as minimal $SU(5)$ and minimal supersymmetric $SU(5)$.