1. Let's start with the spin states of a silver atom. As discussed in class, silver atoms have only two independent spin states, so in this problem we are dealing with a two-dimensional Hilbert space. However, measuring the atom's magnetic moment $\mathbf{m}$ in different directions gives us different bases for this Hilbert space: $(|Z+\rangle, |Z-\rangle)$ are states of definite $m_z = \pm m_B$ (where $m_B = e\hbar/2M_e c$ is the Bohr magneton), $(|X+\rangle, |X-\rangle)$ are states of definite $m_x = \pm m_B$, $(|Y+\rangle, |Y-\rangle)$ are states of definite $m_y = \pm m_B$, and likewise for the more general directions of the magnetic moment.

   Interpreting the Stern–Gerlach experiments with multiple magnetic gaps in terms of the Born rule of quantum probabilities tells us that

$$\begin{aligned} \langle X+|X-\rangle &= \langle Y+|Y-\rangle &= \langle Z+|Z-\rangle &= 0, \\ |\langle Z\pm|X\pm\rangle|^2 &= |\langle Z\pm|Y\pm\rangle|^2 &= |\langle X\pm|Y\pm\rangle|^2 &= \tfrac{1}{2}. \end{aligned} \tag{1}$$

   (a) Use eqs. (1) to show that after a physically-irrelevant change of the overall phases of the ket-vectors $|Z\pm\rangle$, $|X\pm\rangle$, and $|Y\pm\rangle$, the six quantum states become related to each other as

$$|X\pm\rangle = \sqrt{\tfrac{1}{2}}\,|Z+\rangle \pm \sqrt{\tfrac{1}{2}}\,|Z-\rangle\,, \quad |Y\pm\rangle = \sqrt{\tfrac{1}{2}}\,|Z+\rangle \pm i\sqrt{\tfrac{1}{2}}\,|Z-\rangle\,. \tag{2}$$

   (b) Construct the operators $\hat{m}_x$, $\hat{m}_y$, and $\hat{m}_z$ for the three components of the atom's magnetic moment and use eqs. (2) to write down the matrices of those operators in the $(|Z+\rangle, |Z-\rangle)$ basis.

   (c) For an arbitrary direction pointed by a unit vector $\mathbf{n}$, the n^{th}-component of the atom's magnetic moment is $m_n = \mathbf{n}\cdot\mathbf{m}$. Write down the matrix of the the operator $\hat{m}_n = n_x\hat{m}_x + n_y\hat{m}_y + n_z\hat{m}_z$ in the $(|Z+\rangle, |Z-\rangle)$ basis. Calculate the eigenvalues of this matrix and explain the physical meaning of your result.

2. Brush up your knowledge of basic complex analysis. Focus on analytic functions $f(z)$ and contour integrals $\int_\Gamma f(z)\,dz$ over generic contours Γ in the complex plane; make sure you understand which deformations of the contour Γ leave the integral invariant and which do not.

   If you have never studied complex analysis before, a good introductory textbook is *Complex Variables* in the *Schaum Outlines* series, by Spiegel, Lipschutz, Schiller, and Spellman; the PMA library has a few copies. For the purposes of this homework focus on chapter 4; then read the rest of the book when you have time.

   As a simple test of your knowledge, make sure you understand why for any complex α with $\operatorname{Re}\alpha > 0$ and any complex β,

$$\int\limits_{\substack{\text{real}\\ \text{axis}}} \exp\bigl(-\alpha(z-\beta)^2\bigr)\,dz \;=\; \sqrt{\frac{\pi}{\alpha}} \tag{3}$$

   where the integration contour is the whole real axis from $-\infty$ to $+\infty$.

3. Consider a one-dimensional quantum particle with a Gaussian wave function

$$\Psi(x) \;=\; Ce^{ax^2+bx} \tag{4}$$

   where a, b and C are some complex numbers. Note that the discussion in class was limited to the case of real $a < 0$ but in this exercise we allow for any complex a with $\operatorname{Re} a < 0$.

   (a) Calculate the norm $\int dx|\Psi(x)|^2$ of this wave function.

   (b) Calculate the momentum-space wave function $\widetilde{\Psi}(p)$ and show that it also has a Gaussian form

$$\widetilde{\Psi}(p) \;=\; \widetilde{C}e^{\tilde{a}p^2+\tilde{b}p} \tag{5}$$

   for some parameters $\tilde{a}$, $\tilde{b}$, and $\widetilde{C}$. Write down these parameters as explicit functions of a, b, and C.

(c) Verify that the momentum-space and the coordinate-space wave functions have the same norm.

(d) Calculate the expectation values $\langle x\rangle$, $\langle p\rangle$ and the uncertainties Δx and Δp.

(e) Show that for any Gaussian wave function $\Delta X \cdot \Delta P \geq \hbar/2$ and that the equality is achieved whenever a is real.

(f) The Hamiltonian of a free non-relativistic particle is

$$\hat{H} \;=\; \frac{\hat{p}^2}{2M}\,, \tag{6}$$

so in the momentum space

$$\hat{H}\widetilde{\Psi}(p) \;=\; \frac{p^2}{2M} \times \widetilde{\Psi}(p). \tag{7}$$

Solve the time-dependent Schrödinger equation in the momentum space, and show that if the momentum-space wave function has form (5) at time $t = 0$, then it also has form (5) at any time t, albeit for some different parameters $\tilde{a}(t)$, $\tilde{b}(t)$, and $\widetilde{C}(t)$.

Note that this immediately implies that if the coordinate-space wave function has form (4) at time $t = 0$, then it has form (4) at any time t, albeit for time-dependent parameters $a(t)$, $b(t)$, and $C(t)$.

(g) Evaluate the time-dependence of the expectation values $\langle x\rangle$, $\langle p\rangle$ and the uncertainties Δx and Δp and explain the physical meaning of your results.

For simplicity, assume real $a_0 < 0$ at the initial time $t = 0$.