1. Consider the evolution operator $\hat{U}(t, t_0)$ of a quantum particle. The coordinate-basis matrix of this operator

$$U(\mathbf{x}_1, t_1; \mathbf{x}_0, t_0) = \langle \mathbf{x}_1 | \hat{U}(t_1, t_0) | \mathbf{x}_0 \rangle \tag{1}$$

is called the *evolution kernel* or the *propagation amplitude* from $\mathbf{x}_0$ to $\mathbf{x}_1$ in time $t_1 - t_0$. This kernel describes the time evolution of the particle's wave-function as

$$\psi(\mathbf{x}_1, t_1) = \int d^3\mathbf{x}_0 U(\mathbf{x}_1, t_1; \mathbf{x}_0, t_0) \times \psi(\mathbf{x}_0, t_0). \tag{2}$$

Now consider a free non-relativistic spinless particle with Hamiltonian

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2M}. \tag{3}$$

(a) Show that the evolution kernel for this particle is

$$U(\mathbf{x}_1, t_1; \mathbf{x}_0, t_0) = \left(\frac{M}{2\pi i\hbar(t_1 - t_0)}\right)^{3/2} \times \exp\left(\frac{i}{\hbar}\,\frac{m(\mathbf{x}_1 - \mathbf{x}_0)^2}{2(t_1 - t_0)}\right). \tag{4}$$

(b) Spell out the unitarity conditions $\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = 1$ in terms of the evolution kernel (1), specifically in terms of integrals of the form $\int d^3\mathbf{x}\, U^*(\cdots)U(\cdots)$.

(c) Now verify these conditions by explicit integration.

2. A classical charged particle in a magnetic field has *canonical momentum*

$$\mathbf{p} = m\mathbf{v} + \frac{Q}{c}\mathbf{A}(\mathbf{x}_{\text{particle}}) \tag{5}$$

which is quite different from the usual *kinematic momentum* $\vec{\pi} = m\mathbf{v}$, and its classical Hamiltonian is

$$H(\mathbf{x}, \mathbf{p}) = \frac{\vec{\pi}^2}{2m} = \frac{1}{2m}\left(\mathbf{p} - \frac{Q}{c}\mathbf{A}(\mathbf{x})\right). \tag{6}$$

In quantum mechanics, it's the canonical momentum operators $\hat{p}_i$ $(i = x, y, z)$ which obeys the usual commutation relations

$$[\hat{x}_i, \hat{x}_j] \;=\; 0, \quad [\hat{p}_i, \hat{p}_j] \;=\; 0, \quad [\hat{x}_i, \hat{p}_j] \;=\; i\hbar\delta_{ij} \tag{7}$$

so in the coordinate basis they act as $\hat{p}_i \psi(\mathbf{x}) = -i\hbar(\partial/\partial x_i)\psi(\mathbf{x})$. On the other hand, the kinematic momenta

$$\hat{\pi}_i \stackrel{\text{def}}{=} \hat{p}_i \;-\; \frac{Q}{c}\, A_i(\hat{x}, \hat{y}, \hat{z}) \tag{8}$$

act in a more complicated fashion and obey more complicated commutation relations

$$[\hat{x}_i, \hat{x}_j] \;=\; 0, \quad [\hat{x}_i, \hat{\pi}_j] \;=\; i\hbar\delta_{ij}\,, \quad [\hat{\pi}_i, \hat{\pi}_j] \;=\; \frac{i\hbar Q}{c}\, \epsilon_{ijk} B_k(\hat{x}, \hat{y}, \hat{z}). \tag{9}$$

Finally, the Hamiltonian operator $\hat{H}$ follows from the classical Hamiltonian (6) as

$$\hat{H} \;=\; \frac{\hat{\vec{\pi}}^2}{2m} \;=\; \frac{1}{2m}\left(\hat{\mathbf{p}} \;-\; \mathbf{A}(\hat{\mathbf{x}})\right)^2. \tag{10}$$

(a) Unless you have attended my extra lecture on September 9, read the parts of my notes on classical mechanics and canonical quantization where I explain eqs. (5) through (10). Specifically, pages 5–7 where I explain the classical mechanics of a charged particle, and pages 8–9 where I explain the commutation relations.

(b) Use the commutation relations (9) to derive the Ehrenfest equations for the quantum charged particle. Specifically, show that

$$\frac{d}{dt}\left\langle \hat{\mathbf{x}} \right\rangle \;=\; \frac{1}{m}\left\langle \hat{\vec{\pi}} \right\rangle \quad \text{and} \quad \frac{d}{dt}\left\langle \hat{\vec{\pi}} \right\rangle \;=\; \frac{Q}{2mc}\left\langle \hat{\vec{\pi}} \times \hat{\mathbf{B}} - \hat{\mathbf{B}} \times \hat{\vec{\pi}} \right\rangle \tag{11}$$

where $\hat{\mathbf{B}} \stackrel{\text{def}}{=} \mathbf{B}(\hat{x}, \hat{y}, \hat{z})$.

(c) Now let's subject the particle to both electric and magnetic fields and allow both fields to be time-dependent, thus time-dependent Hamiltonian

$$\hat{H}(t) \;=\; Q\Phi(\hat{\mathbf{x}},t) \;+\; \frac{1}{2m}\left(\hat{\mathbf{p}} \;-\; \mathbf{A}(\hat{\mathbf{x}},t)\right)^2 \tag{12}$$

Show that in this case, the Ehrenfest equations become

$$\frac{d}{dt}\left\langle \hat{\mathbf{x}} \right\rangle \;=\; \frac{1}{m}\left\langle \hat{\vec{\pi}} \right\rangle \quad \text{and} \quad \frac{d}{dt}\left\langle \hat{\vec{\pi}} \right\rangle \;=\; Q\left\langle \hat{\mathbf{E}} \right\rangle \;+\; \frac{Q}{2mc}\left\langle \hat{\vec{\pi}} \times \hat{\mathbf{B}} - \hat{\mathbf{B}} \times \hat{\vec{\pi}} \right\rangle . \tag{13}$$

Hint: use Heisenberg–Dirac equations for the time-dependent operators, and remember that in a time-dependent vector potential $\mathbf{A}(\mathbf{x},t)$, the kinematic momentum operator (8) becomes *explicitly* time-dependent,

$$\frac{\partial}{\partial t}\,\hat{\vec{\pi}} \;=\; -\frac{Q}{c}\,\frac{\partial \mathbf{A}}{\partial t}(\hat{\mathbf{x}},t). \tag{14}$$

3. Finally, a refresher of undergraduate-level theory of orbital angular momentum

$$\hat{\mathbf{L}} \;\stackrel{\text{def}}{=}\; \hat{\mathbf{x}} \times \hat{\mathbf{p}}, \qquad i.\ e.,\ \ \hat{L}_i \;\stackrel{\text{def}}{=}\; \epsilon_{ijk}\hat{x}_j\hat{p}_k\,. \tag{15}$$

It is also a good drill for the use of the canonical commutation relations

$$[\hat{x}_i,\hat{x}_j] = 0, \quad [\hat{p}_i,\hat{p}_j] = 0, \qquad [\hat{x}_i,\hat{p}_j] = i\hbar\delta_{ij}\,. \tag{16}$$

Without using coordinate-basis or any other wave functions, use the relations (16) to show that:

(a) $[\hat{x}_i,\hat{L}_j] = i\hbar\epsilon_{ijk}\hat{x}_k\,$;

(b) $[\hat{p}_i,\hat{L}_j] = i\hbar\epsilon_{ijk}\hat{p}_k\,$;

(c) $[\hat{L}_i,\hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$ and therefore $\hat{\mathbf{L}} \times \hat{\mathbf{L}} = i\hbar\hat{\mathbf{L}}$;

(d) $[\hat{\mathbf{p}}^2,\hat{\mathbf{L}}] = 0 = [f(\hat{r}),\hat{\mathbf{L}}]$ for any function f of the radius $\hat{r} = (\hat{\mathbf{x}}^2)^{1/2}$;

(e) $[\hat{\mathbf{L}}, \hat{\mathbf{L}}^2] = 0$;

(f) $\hat{\mathbf{p}}^2 = \hat{p}_r^2 + \hat{r}^{-2}\hat{\mathbf{L}}^2$, where $\hat{p}_r \stackrel{\text{def}}{=} \frac{1}{2}\{\widehat{\frac{x_i}{r}}, \hat{p}_i\}$ (note: $\hat{p}_r$ so defined is hermitian).

Now, let us make use of the coordinate-basis wave functions.

(g) Given a wave-function $\Psi(r, \theta, \phi) = \langle r, \theta, \phi | \Psi \rangle$ in the spherical-coordinate basis, compute $\langle r, \theta, \phi | \, \hat{p}_r \, | \Psi \rangle$ and $\langle r, \theta, \phi | \, \hat{p}_r^2 \, | \Psi \rangle$.

(h) Finally, use the above results to calculate $\langle r, \theta, \phi | \, \hat{\mathbf{L}}^2 \, | \Psi \rangle$.
Hint: In the coordinate basis $\hat{\mathbf{p}}^2 = -\hbar^2 \nabla^2$; look up in any E&M textbook how the Laplacian ∇^2 acts in spherical coordinates, then use (f) and (g).