

1. Consider a spinless charged particle in a uniform magnetic field \mathbf{B} . For simplicity, assume that the particle is free to move in the xy plane \perp to the magnetic field's direction, but in the z direction along the field. As we saw in the [previous homework](#) (set#4, problem 2), the Hamiltonian operator for this particle is

$$\hat{H} = \frac{\hat{\pi}_x^2 + \hat{\pi}_y^2}{2M} \quad (1)$$

where

$$\begin{aligned} [\hat{x}, \hat{y}] &= 0, \\ [\hat{x}_i, \hat{\pi}_i] &= i\hbar\delta_{ij} \quad (\text{for } i, j = x, y), \\ [\hat{\pi}_x, \hat{\pi}_y] &= i\frac{QB\hbar}{c}. \end{aligned} \quad (2)$$

(a) Let

$$\hat{a} = \sqrt{\frac{c}{2\hbar|QB|}} (\hat{\pi}_x + i \text{sign}(QB) \hat{\pi}_y) \quad (3)$$

and show that this non-Hermitian operator obeys $[\hat{a}, \hat{a}^\dagger] = 1$.

- (b) Rewrite the Hamiltonian (1) in terms of the \hat{a} and \hat{a}^\dagger operators, then show that its spectrum consists of discrete *Landau levels*

$$E_n = \hbar\Omega(n + \frac{1}{2}), \quad n = 0, 1, 2, \dots \quad (4)$$

where

$$\Omega = \frac{|QB|}{Mc} \quad (5)$$

is the classical *cyclotron frequency* of the particle moving in a Larmor circle in the magnetic field.

- (c) Show that for a classical particle moving in a Larmor circle, the circle's center is located at

$$x_c = x + \frac{c}{QB} \pi_y, \quad y_c = y - \frac{c}{QB} \pi_x. \quad (6)$$

- (d) Show that the quantum analogues \hat{x}_c and \hat{y}_c of the center's coordinate commute with both $\hat{\pi}_x$ and $\hat{\pi}_y$ and hence with the Hamiltonian \hat{H} . In other words, both \hat{x}_c and \hat{y}_c are conserved operators.

- (e) Show that the \hat{x}_c and \hat{y}_c do not commute with each other; instead

$$[\hat{x}_c, \hat{y}_c] = \frac{-i\hbar c}{QB}. \quad (7)$$

- (f) Use the commutator (7) to show that each Landau energy level is infinitely degenerate. Hint: build Harmonic-oscillator-like operators \hat{b} and \hat{b}^\dagger with $[\hat{b}, \hat{b}^\dagger] = 1$ from \hat{x}_c and \hat{y}_c , then show that an entire infinite tower of oscillator-like states must exist at every Landau level.

2. Now let's learn about the *coherent states* of a harmonic oscillator.

- (a) First, a lemma about functions of \hat{a} or \hat{a}^\dagger operators. Let $f(\xi)$ be any analytic function of a complex number ξ and $f'(x) = df/d\xi$ its derivative. Show that

$$[\hat{a}, f(\hat{a}^\dagger)] = f'(\hat{a}^\dagger) \quad \text{and} \quad [\hat{a}^\dagger, f(\hat{a})] = -f'(\hat{a}). \quad (8)$$

Next, for any complex number ξ we define the coherent state $|\xi\rangle$ as

$$|\xi\rangle \stackrel{\text{def}}{=} e^{-|\xi|^2/2} \exp(\xi \hat{a}^\dagger) |0\rangle \quad (9)$$

where $|0\rangle = |n=0\rangle$ is the oscillator's ground state.

- (b) Calculate $\langle n|\xi\rangle$ for all $n = 0, 1, 2, \dots$, then show that the state (9) is normalized, *i.e.* $\langle \xi|\xi\rangle = 1$.

The operators \hat{a} and \hat{a}^\dagger cannot be diagonalized. However, \hat{a} has an eigen-ket for any complex eigenvalue while \hat{a}^\dagger has an eigen-bra for any complex eigenvalue. On the other hand, \hat{a}^\dagger has no eigen-kets at all, while \hat{a} has no eigen-bras.

- (c) Show that the coherent state $|\xi\rangle$ is the eigen-ket of \hat{a} for the eigenvalue ξ ; likewise, $\langle\xi|$ is an eigen-bra of \hat{a}^\dagger for the eigenvalue ξ^* :

$$\hat{a}|\xi\rangle = \xi|\xi\rangle, \quad \langle\xi|\hat{a}^\dagger = \xi^*\langle\xi|. \quad (10)$$

Hint: use part (a) to show that $\hat{a}\exp(\xi\hat{a}^\dagger) = \exp(\xi\hat{a}^\dagger)(\hat{a} + \xi)$, then apply both sides of this equation to $|0\rangle$.

- (d) Show that \hat{a}^\dagger has no eigen-kets for any complex eigenvalues while \hat{a} has no eigen-bras.
Hint: show that if $\hat{a}^\dagger|\psi\rangle = \lambda|\psi\rangle$ then $|\psi\rangle$ is un-normalizable because $|\langle n|\psi\rangle|^2$ increases with n .

Coming back to the coherent states, in any coherent state ξ , the expectation value of any *normal-ordered* product of raising and lowering operators — *i.e.*, a product $(\hat{a}^\dagger)^m(\hat{a})^n$ in which all raising operators are to the left of all the lowering operators — is simply

$$\langle\xi|(\hat{a}^\dagger)^m(\hat{a})^n|\xi\rangle = \xi^{*m}\xi^n. \quad (11)$$

- (e) Prove this.

A coherent state $|\xi\rangle$ does not have a definite energy (except for $\xi = 0$). However, for the highly excited coherent state with $\langle E\rangle \gg \hbar\omega$, the *relative* energy uncertainty becomes small, $\Delta E \ll \langle E\rangle$.

- (f) Calculate $\langle E\rangle$ and ΔE in a coherent state ξ .

Hint: prove and use $\hat{n}^2 = (\hat{a}^\dagger)^2(\hat{a})^2 + \hat{n}$, then use eq. (11).

Since the operator \hat{a} is not Hermitian, its eigen-kets are not orthogonal to each other. Nevertheless, the overlap between 2 coherent states $|\xi\rangle$ and $|\eta\rangle$ becomes exponentially small for large $|\xi - \eta|$.

- (g) Calculate the overlap and show that $|\langle\eta|\xi\rangle|^2 = \exp(-|\xi - \eta|^2)$.

- (h) Finally, show that the set of all the coherent states $|\xi\rangle$ (for all complex ξ) forms an over-complete basis of the harmonic oscillator's Hilbert space,

$$\int \frac{d^2\xi}{\pi} |\xi\rangle \langle\xi| = \hat{1} \quad (12)$$

where $d^2\xi = d(\Re\xi)d(\Im\xi) = |\xi| d(|\xi|)d(\arg \xi)$.

Hint: calculate the matrix elements $\langle m|$ integral (12) $|n\rangle$.

3. Finally, consider the dynamics of coherent states. If the initial state of a harmonic oscillator is coherent, then it remain a coherent state at all future times, but for a time-dependent $\xi(t)$, namely

$$\xi(t) = \xi_0 \times e^{-i\omega t}. \quad (13)$$

- (a) Show that the state

$$|\psi\rangle(t) = e^{-i\omega t/2} |\xi(t)\rangle \quad (14)$$

where $\xi(t)$ evolves according to eq. (13) obeys the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi\rangle(t) = \hat{H} |\psi\rangle(t). \quad (15)$$

- (b) Calculate the expectation values $\langle \hat{q} \rangle$ and $\langle \hat{p} \rangle$ of the position and momentum in a coherent state ξ . Then show that when $\xi(t)$ evolves according to eq. (13), these expectation values obey the classical equations of motion.
- (c) Calculate the uncertainties Δq and Δp in a coherent state and show that $\Delta q \times \Delta p = \frac{1}{2}\hbar$, the minimum allowed by the Heisenberg's uncertainty principle.

In an earlier homework [set#1](#) we saw that that the Heisenberg bound is saturated by the Gaussian wave packets (with real coefficients of $-x^2$ in the exponent). The coherent state also saturate this bound because they are Gaussian wave packets of this kind.

(d) Solve the equation $(\hat{a} - \xi)|\psi\rangle$ in the coordinate basis and show that the solution is indeed the Gaussian wave packet

$$\psi(q) = C \times \exp\left(-\frac{m\omega}{2\hbar} \times (q - \bar{q})^2 + \frac{i\bar{p}}{\hbar} \times q\right) \quad (16)$$

where $\bar{q} = \langle \xi | \hat{q} | \xi \rangle$, $\bar{p} = \langle \xi | \hat{p} | \xi \rangle$, and C is a constant overall factor.

Note: the magnitude of the constant C obtains from the normalization condition $\langle \psi | \psi \rangle = 1$, but determining the phase of C takes extra information, for example requiring $|\psi\rangle = |\xi\rangle$ having exactly the same overall phase as in eq. (9). The correct answer is

$$C = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \times e^{-i\bar{p}\bar{q}/2\hbar}, \quad (17)$$

but deriving this formula is **not** a part of this homework assignment.

The bottom line is, the best way to see the near-classical oscillations in quantum mechanics is to look at the coherent states $|\xi\rangle$ with $\xi(t) = \xi_0 e^{i\omega t}$. These states provide for minimal uncertainties Δq and δp while the expectation values $\langle q \rangle(t)$ and $\langle p \rangle(t)$ oscillate in a classical manner. Also, while the coherent states do not have definite energies, the *relative* energy uncertainty becomes small for the highly excited states (*cf.* problem 2(e)).

By comparison, the stationary states $|n\rangle$ do not show any classical-like motion. Indeed, not only there is no motion at all in a stationary state, but also

$$\langle n | \hat{q} | n \rangle = \langle n | \hat{p} | n \rangle = 0 \quad (18)$$

while the uncertainties grow with n :

$$(\Delta q)^2 = \frac{\hbar}{2m\omega} \times (2n+1), \quad (\Delta p)^2 = \frac{\hbar\omega m}{2} \times (2n+1) \quad \implies \quad \Delta q \times \Delta p = \hbar \times \left(\frac{1}{2} + n\right). \quad (19)$$

(e) Verify all these formulae.