1. First, a reading assignment: my notes on the saddle point method and on Airy functions.

2. In class I argued that for the highly-excited bound states of a potential well, the WKB approximation for the wave functions should become accurate,

$$\Psi_n^{\rm WKB}(x)\ \to\ \Psi_n^{\rm true}(x)\quad\text{for } n\to\infty. \tag{1}$$

In this problem, we shall verify this rule for the harmonic oscillator.

(a) Write down the WKB approximation for the oscillator's eigenstates $\Psi_n(x)$. Don't bother with the overall normalization of the $\Psi_n^{\rm WKB}(x)$ wave-functions, but please describe them in both classically-allowed and classically-forbidden regions of space.

For future convenience, rescale the coordinate x to

$$y\ =\ \frac{x}{\text{classical oscillation amplitude}}\ =\ \sqrt{\frac{m\omega}{(2n+1)\hbar}}\times x, \tag{2}$$

so the classical turning points are at $y=\pm1$.

Back in homework set#2 (problem#3), you (should have) proved a Lemma about the exact wave functions of the harmonic oscillator:

$$\Psi_n^{\rm true}(x)\ =\ \text{const}\times\exp\left(+\frac{m\omega x^2}{2\hbar}\right)\times\int\limits_{-\infty}^{+\infty}dk\,k^n\times\exp\left(ikx\,-\,\frac{\hbar k^2}{4m\omega}\right). \tag{3}$$

(b) Rescale x and k variables and rewrite eq. (3) as

$$\Psi_n^{\rm true}(y)\ =\ \text{const}\times\int\limits_{-\infty}^{+\infty}dq\,f(q)\times\exp\bigl((2n+1)g(q,y)\bigr) \tag{4}$$

for some n–independent analytic functions $f(q)$ and $g(q,y)$.

(c) Now take the $n \to \infty$ limit of the integral (4) using the saddle-point method explained in my notes on the subject. Show that for both classically allowed region $-1 < y < +1$ and for the classically forbidden regions $y < -1$ or $y > +1$,

$$\Psi_n^{\rm true}(y) \ \xrightarrow[n\to\infty]{} \ \Psi_n^{\rm WKB}(y). \tag{5}$$

Hints: The exponent $g(q)$ in the integral (4) has two complex saddle points. For $-1 < y < +1$, both of these saddle point contribute to the large n limit of the integral. But for $y > +1$ or $y < -1$, only one saddle point contributes and the other does not. To find which saddle point is which, check the directions of complex contours that traverse that point as a mountain highway crosses a pass, up from a valley and then down to another valley; if a direction parallel to real axis is bad, this saddle point does not contribute.

3. Finally, a simple exercise of the (corrected) Bohr–Sommerfeld quantization rule: for a bound state, the action over one full period must be a half-integral multiple of the Planck constant $2\pi\hbar$,

$$\oint p(t)\,dx(t) \ = \ 2\pi\hbar\times(n+\tfrac{1}{2}), \quad n=0,1,2,3,\ldots. \tag{6}$$

Consider the radial motion of an electron in a hydrogen-like atom or ion. Between the Coulomb potential and the centrifugal potential due to angular momentum, the net effective potential for the radial motion is

$$V_{\rm eff}(r) \ = \ -\frac{Ze^2}{r} \ + \ \frac{\mathbf{L}^2}{2mr^2}\,. \tag{7}$$

(a) Find the classical turning points r_1 and r_2 for a bound state with $E < 0$.

(b) Calculate the classical action for one full period of the radial motion.

Mathematical hint:

$$\int\limits_A^B dx\,\frac{\sqrt{(B-x)(x-A)}}{x} \ = \ \frac{\pi}{2}\bigl(A+B-2\sqrt{AB}\bigr). \tag{8}$$

(c) Apply the Bohr–Sommerfeld rule (6) and show that it yields the correct bound state energies provided we approximate

$$\mathbf{L}^2 \ = \ \hbar^2\ell(\ell+1) \ \approx \ \hbar^2(\ell+\tfrac{1}{2})^2. \tag{9}$$