PHY–389K: Quantum Mechanics. Section of Professor Vadim Kaplunovsky.
Homework set #10. Due November 14, 1996.

1. Covariant Derivatives

For a charged particle with a wave function $\Psi(\mathbf{x},t)$, the probability density and the probability current are respectively

$$\rho_p(\mathbf{x},t) \stackrel{\text{def}}{=} |\psi|^2, \qquad \mathbf{J}_p(\mathbf{x},t) \stackrel{\text{def}}{=} \frac{\hbar}{M}\,\mathrm{Im}\left(\psi^*\mathbf{D}\psi\right) \tag{1.1}$$

where $\mathbf{D}$ is the gauge-covariant gradient. These density and current are obviously gauge invariant; they also satisfy the continuity equation

$$\nabla\cdot\mathbf{J}_p + \frac{\partial\rho_p}{\partial t} = 0, \tag{1.2}$$

but this is not so obvious.

- ∗ First, prove a lemma:

$$\nabla\cdot\left(\psi^*\mathbf{D}\psi\right) = \left|\mathbf{D}\psi\right|^2 + \psi^*\mathbf{D}^2\psi\,. \tag{1.3}$$

- Second, use this lemma to verify the continuity equation (1.2).

Next, consider a spin-half charged particle such as electron. Starting with the Dirac equation for a free relativistic electron, making a minimal substitution $\partial_\mu \to D_\mu$ and then taking the non-relativistic limit, one arrives at

$$i\hbar D_t\psi = -\frac{\hbar^2}{2M}\left(\vec{\sigma}\cdot\mathbf{D}\right)^2\psi \tag{1.4}$$

where $\sigma_{1,2,3}$ are the Pauli matrices for the electron's spin.

⋆ Show that this covariant Schrödinger equation corresponds to the Pauli Hamiltonian

$$\hat{H}_{\rm Pauli} \;=\; \frac{1}{2M}\,\hat{\vec{\pi}}^{\,2} \;-\; e\hat{\Phi} \;+\; \frac{eg_e}{2Mc}\,\hat{\mathbf{S}}\cdot\hat{\mathbf{B}} \tag{1.5}$$

where $\hat{\mathbf{S}} = (\hbar/2)\vec{\sigma}$ is the electron's spin and $g_e = 2$ its gyromagnetic factor.

2. Superconductivity

Superconductors are macroscopic systems that behave in some essentially quantum ways; many useful devices such as very sensitive magnetometers (SQUIDs) are based on such quantum features. The microscopic theory of superconductivity is quite complicated and took many years to develop; however, the macroscopic theory of superconductivity is much easier. The goal of these notes (and the exercises contained in them) is to give you a basic understanding of some of the phenomena involved.

The basic fact of superconducting life is that in cold, non-magnetic metals a small fraction of the electrons form so-called Cooper pairs. The two electrons forming a pair have momenta very close to the Fermi surface and almost exactly opposite to each other; the spins of the two electrons are also opposite. Thus on the whole a Cooper pair is a slowly-moving spinless boson of electric charge $-2e$; it is the presence of these charged bosons that gives rise to superconductivity. Or rather, it's the Bose–Einstein condensate of the Cooper pairs which gives rise to the superconductivity. Indeed, the Cooper pairs hardly exist outside this condensate: the excitations of the superconductor's ground state break the pairs into individual electrons rather than kick a pair out of the condensate but keep it unbroken.

At the phenomenological level, we may describe the BEC of Cooper pairs by the Landau–Ginzburg complex classical field $\Psi(\mathbf{x},t)$ obeying Schrödinger-like non-linear equation

$$i\hbar D_t\Psi \;=\; -\,\frac{\hbar^2}{2M}\,\mathbf{D}^2\Psi +\; \left(\lambda|\Psi|^2-\mu\right)\Psi \tag{2.1}$$

where $M\neq 2m_e$ is the effective mass of a Cooper pair, μ is the chemical potential, and λ parametrizes the short-distance repulsive forces between the Cooper pairs. *Heuristically,* we may think of the BEC as having all Cooper pairs being in the same single-particle quantum state with a wave function $\psi(\mathbf{x},t)$; in terms of this wave-function, the Landau–Ginzburg field is simply

$$\Psi(\mathbf{x},t) \;=\; \sqrt{N_{\text{pairs}}}\times\psi(\mathbf{x},t), \tag{2.2}$$

where the $\sqrt{N_{\text{pairs}}}$ factor makes $n_s = |\Psi|^2$ the local density of the Cooper pair condensate. The non-linear term in the field equation (2.1) stems from the Mean Field Theory approximation to the interactions between the pairs. That is, we neglect the rather weak interactions between individual pairs, but the collective effect of all the other pairs on any one pair gives rise to an effective potential

$$\mathcal{V}(\mathbf{x},t) \;\approx\; \lambda|\Psi(\mathbf{x},t)|^2 \;-\; \mu. \tag{2.3}$$

Combining this mean-field effective potential with the macroscopic electric and magnetic forces on a charged Cooper pair gives rise to the Schrödinger equation

$$i\hbar D_t\psi \;=\; -\frac{\hbar^2}{2M}\,\mathbf{D}^2\psi \;+\; \mathcal{V}\psi \tag{2.4}$$

for the wave function of a pair, and hence eq. (2.1) for the Landau–Ginzburg field.

Going beyond this heuristic explanation involves a Hilbert of an arbitrary number of Cooper pairs, creation and annihilation operators for the pairs, and ultimately the non-relativistic quantum field theory. The classical limit of that quantum field is the Landau–Ginzburg field. I have outlined how this works for the superfluid liquid helium — or a BEC condensate of heavy atoms — in an extra lecture on 10/1. The Landau–Ginzburg theory of the BEC condensate of Cooper pairs works in a similar manner, except for the electric charge $q = -2e$ of a Cooper pair calls for the covariant derivatives D_t and $\mathbf{D}$ in eq. (2.1).

Electric charges and currents

For a single charge particle like a Cooper pair, the probability density and the probability current are as in eq. (1.1), for for the BEC condensate of the pairs we simply multiply by

$N_{\rm pairs}$ to get the number density n_s and the number flux $\vec{\mathcal{F}}_s$, thus

$$n_s \;=\; N\,|\psi|^2 \;=\; \Psi|^2 \quad \text{and} \quad \vec{\mathcal{F}} \;=\; N\,\frac{\hbar}{M}\,\text{Im}(\psi^*\mathbf{D}\psi) \;=\; \frac{\hbar}{M}\,\text{Im}(\Psi^*\mathbf{D}\Psi). \tag{2.5}$$

Consequently, the superconducting charge density and the electric current density are, respectively,

$$\rho_s \;=\; -2en_s \;=\; -2e|\Psi|^2 \quad \text{and} \quad \mathbf{J}_s \;=\; -2e\vec{\mathcal{F}}_s \;=\; \frac{-2e\hbar}{M}\,\text{Im}(\Psi^*\mathbf{D}\Psi). \tag{2.6}$$

However, the superconducting BEC condensate of Cooper pairs is not the only charged ingredient in a superconductor, there is also a Fermi gas of normal (non-superconducting) electrons and the lattice of ion cores, thus

$$\begin{aligned} \rho_{\rm net} \;&=\; \rho_{\rm ion} \;+\; \rho_n \;+\; \rho_n\,, \\ \mathbf{J}_{\rm net} \;&=\; \mathbf{J}_n \;+\; \mathbf{J}_{\mathbf{s}}\,. \end{aligned} \tag{2.7}$$

Moreover, in all practical situations the net electric charge density in a superconducting metal is zero, $\rho_{\rm net} = 0$. On the other hand, the normal and the superconducting currents do not cancel each other. Instead, as long as superconductivity exists, the supercurrent $\mathbf{J}_s$ flows without resistance and shorts out the electric field $\mathbf{E} \to 0$, so by the Ohm's law $\mathbf{J}_n = \sigma\mathbf{E}$, the normal current does not flow. So the bottom line is, in a superconductor

$$\rho_{\rm net} \;=\; 0 \quad \text{but} \quad \mathbf{J}_{\rm net} \;=\; \mathbf{J}_s\,. \tag{2.8}$$

- Let's describe the complex Landau–Ginzburg field in terms of its magnitude and phase as

$$\Psi(\mathbf{x},t) \;=\; \sqrt{n_s(\mathbf{x},t)}\,\exp(iS(\mathbf{x},t)/\hbar). \tag{2.9}$$

  Show that in terms of these variables, the supercurrent becomes

$$\mathbf{J}_s(\mathbf{x},t) \;=\; \frac{-2en_s}{M}\left(\nabla S(\mathbf{x},t) \;+\; \frac{2e}{c}\mathbf{A}(\mathbf{x},t)\right). \tag{2.10}$$

The explicit presence of the vector potential in this formula gives rise to some rather spectacular effects. In particular, the magnetic field $\vec{B}$ cannot penetrate a bulk superconductor much beyond a certain depth. This is the *Meissner's effect* and it's exhibited by all superconductors in weak magnetic fields; strong magnetic fields destroy the superconductivity.

- Assume uniform $n_s \neq 0$ for a bulk superconductor and a time-independent magnetic field $\mathbf{B}(\mathbf{x})$. Use Maxwell's equations together with eq. (2.10) for the supercurrent and show that the magnetic field in the superconductor obeys

$$\left(\vec{\nabla}^2 - \ell^{-2}\right)\mathbf{B}(\mathbf{x}) \;=\; 0 \tag{2.11}$$

  and hence cannot penetrate the superconductor to a depth much beyond the so-called *Landau depth* ℓ. Compute ℓ in terms of Cooper pair density n and whatever constants you may need.

The Meissner effect leads to many other interesting phenomena, such as magnetic flux quantization. Indeed, consider a closed loop of superconducting wire: If the wire is thick enough to expel the magnetic field from its interior, the supercurrent would also be expelled from the wire's interior and flow through the wire's skin only. Hence, in the wire's interior $\nabla S + \frac{2e}{c}\mathbf{A} = 0$; integrating this equation along the wire's centerline gives us

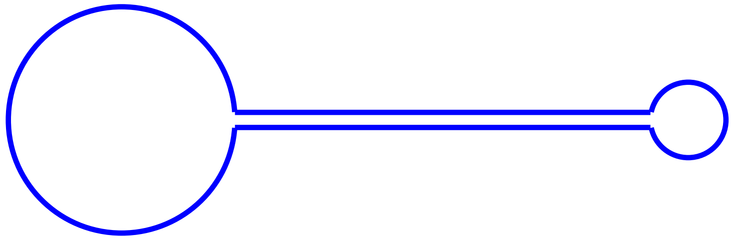
$$\oint\limits_{\text{wire}} \mathbf{A}\cdot d\mathbf{x} \;=\; -\frac{c}{2e}\,\Delta S. \tag{2.12}$$

The left hand side of this equation is the magnetic flux F through the wire loop. The right hand side of eq. (2.12) involves the accumulated change of the phase $S/\hbar$ of the LG field; for a closed loop this total phase change must be an integer multiple of 2π. Therefore, eq. (2.12) tells us that *magnetic flux through a closed loop of a superconducting wire must be an integer multiple of*

$$F_0 \;=\; \frac{2\pi\hbar c}{2e} \tag{2.13}$$

This flux quantization condition is closely related to the Aharonov-Bohm effect.

Magnetic flux quantization is used in superconducting devices such as a magnetic amplifier, which is basically a loop of superconducting wire that looks like



(2.14)

Since the total flux through both loops is quantized, it cannot be changed adiabatically. Therefore, small adiabatic changes of the magnetic field going through the big loop result in much bigger changes of the field in the small loop. The amplification factor is given by (minus) the area ratio.

Josephson junctions and SQUIDS

A Josephson's junction is a week link in a superconducting wire. It can be a sharp point contact between two wires, or a very thin dielectric film separating two thick films of superconducting metal, or some other obstacle through which the Cooper pairs have to tunnel in order to get from one side of the junction to the other.

In the Landau-Ginsburg description, the junction appears as potential barrier: The effective potential $\mathcal{V}(\mathbf{x})$ now acquires an additional term $\Delta\mathcal{V}(\mathbf{x})$ that vanishes in the interior of the superconductor but become positive (and large, albeit finite) in the junction area, thus

$$\mathcal{V}(\mathbf{x},t) \;=\; \lambda|\Psi(\mathbf{x},t)|^2 \;-\; \mu \;+\; \Delta\mathcal{V}(\mathbf{x}) \;=\; \lambda(|\Psi(\mathbf{x},t)|^2 - n_0) \;+\; \Delta\mathcal{V}(\mathbf{x}). \tag{2.15}$$

where $n_0 = \mu/\lambda$ is the Cooper pair density in the bulk superconductor. Consequently, the stationary form of the Landau-Ginsburg equation (2.1) becomes

$$\frac{-\hbar^2}{2M}\,\mathbf{D}^2\Psi(\mathbf{x}) \;+\; \lambda\bigl(|\Psi(\mathbf{x})|^2 - n_0\bigr)\Psi(\mathbf{x}) \;+\; \Delta\mathcal{V}(\mathbf{x})\Psi(\mathbf{x}) \;=\; 0 \tag{2.16}$$

To solve the equation, we consider three distinct zones of space: (A) interior and immediate vicinity of one of the superconducting wires; (B) ditto for the other wire; (C) the barrier between the wires. If the barrier is sufficiently hard to tunnel through, one can solve eq. (2.16) for the interior and immediate vicinity of one wire while totally disregarding the very existence of the other wire. One simply imposes the boundary condition that $\Psi_A \to \sqrt{n_0} e^{i\phi_1}$ as one goes into the first superconductor and $\Psi_A \to 0$ as one goes away from it. In the absence of magnetic field, eq. (2.16) is real, so $\Psi_A(\mathbf{x})$ should be real apart from the overall factor $e^{i\phi_1}$. Similarly, in the interior and immediate vicinity of the second wire we have Ψ_B that is real apart from an overall phase factor $e^{i\phi_2}$. In the third zone — the middle of the barrier — the pair density $n = |\Psi|^2$ is so small that one can safely ignore the non-linear term in eq. (2.16); hence the solution in this zone is simply the algebraic sum of the tails of Ψ_A and Ψ_B.

∗ Show that for $\mathbf{A} = 0$, the Landau–Ginzburg field in the middle of the barrier has general form

$$\Psi_C(\mathbf{x}) \;=\; e^{i\phi_1} \times \Psi_1(\mathbf{x}) \;+\; e^{i\phi_2} \Psi_2(\mathbf{x}) \tag{2.17}$$

for some *real* functions $\Psi_1(\mathbf{x})$ and $\Psi_2(\mathbf{x})$.
Note: you do not need the specific form of these functions, so don't waste your time finding what they look like. In particular, do not try to solve any non-linear differential equations.

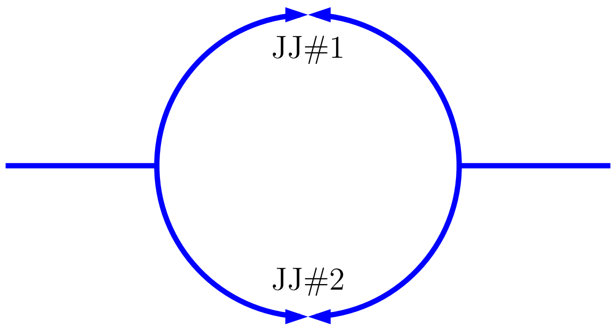
⋆ Use eq. (2.17) to show that in the absence of magnetic field, the total supercurrent through the Josephson's junction is

$$I \;=\; I_0 \sin(\phi_1 - \phi_2) \tag{2.18}$$

where $\phi_{1,2}$ are the phases of the LG field in the two superconductors connected by the junction and I_0 is a constant depending on the specifics of the junction's structure.

Experimentally, I_0 is the maximal supercurrent that tunneling Cooper pairs can carry through the junction. Single electrons can carry a bigger electric current, but it would be a normal current, subject to resistance and thus needing a voltage drop. Moreover, according to the microscopic theory of superconductivity developed by Bardeen, Cooper and Schriffer, a normal current cannot flow through a Josephson junction until the voltage drop exceeds some threshold value (typically, a few millivolts). Experimentally, this is indeed the case: As one increases the current through a Josephson's junction, the voltage stays exactly zero until the maximal supercurrent I_0 is reached, then suddenly jumps to a few millivolts; after that, it continues to grow with the current.

SQUIDs are Superconducting QUantum Interferometry Devices. They come in many shapes, but the simplest one consists of two Josephson's junctions in a single loop of superconducting wire:



(2.19)

In the absence of magnetic field, the maximal supercurrent that can flow through a symmetric SQUID is clearly $2I_0$; in the presence of magnetic field things are much more interesting.

- ◇ Show that in the presence of magnetic field, the maximal supercurrent that can flow through the SQUID is

$$I_{\max}(B) \;=\; 2I_0 \left|\cos\frac{\pi F}{F_0}\right| \tag{2.20}$$

  where F is the magnetic flux through the SQUID's loop and F_0 is given by f-la (2.13). Assume that the field is not so strong as to affect the junctions themselves (otherwise, I_0 would also change with the field) but only their interference.

Practically, SQUIDs are used as very sensitive magnetometers: According to eq. (2.20), tiny changes of the magnetic field through the SQUID's loop result in easily measurable changes in the maximal supercurrent $I_{\max}(B)$. And when even higher sensitivity is needed, one may combine a SQUID with a magnetic amplifier, or with a cascade arrangement of amplifiers; the engineering of magnetic couplings between SQUIDs and amplifier loops is tricky, but the physics is quite straightforward.